INVESTIGATION OF STRAPPED-DOWN REFERENCE SYSTEMS

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Bachelor of Science

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1972

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
MASTER OF SCIENCE
July, 1974

INVESTIGATION OF STRAPPED-DOWN

REFERENCE SYSTEMS

Thesis Approved:

Thesis Adviser

Dean of the Graduate College

ACKNOWLEDGMENT

I would like to express my gratitude for the help which people have given.

First, my appreciation to Dr. Henry R. Sebesta, my major adviser and a good friend. Dr. Sebesta arranged for my support and has left me freedom to pursue avenues of my own choosing. I owe him a great deal.

I thank the other members of my committee, Dr. Hermann G. Burchard, Dr. Robert J. Mulholland, and Dr. Larry D. Zirkle for their assistance and their helpful criticism. In particular, I appreciate the help Dr. Burchard has given me with the math.

I appreciate the stimulating discussions with my officemates Lynn Ebbesen, B. N. Murali, M. S. Kahn, Charlie Chan and Gary Kreie.

I acknowledge financial support which I received in the form of a research assistantship from the Air Force Office of Scientific Research, Grant Number AFOSR 73-2458.

I thank Ginny Schulte and Barbara Bush for their help in preparing the draft of this thesis and for the food.

Finally, I thank Lynn Danvers for preparing the final manuscript for publication.

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CHAPTER I

INTRODUCTION

1. Background

This report describes technical results of a study conducted for the Air Force Office of Scientific Research in the subject area of strapped-down reference systems. A motivation for the work is to seek technology applicable to increased accuracies in pointing and tracking systems. Pointing and tracking systems are an inherent aspect of several weapon and non-weapon programs being conducted by the Air Force, NASA, Navy and other agencies at their research and development laboratories. One specific program, for example, is under the direction of the Air Force Weapons Laboratory (AFWL) at Kirtland Air Force Base, New Mexico. In this program, Hughes Aircraft Company, as a major AFWL contractor, has designed, constructed and flight tested a precision pointing and tracking system for airborne applications.

A critical aspect of the pointing and tracking problem is the requirement for measuring and/or predicting the

relative motion of the target being tracked. Typically, the approaches currently being recommended and implemented include high-speed digital computers. One function of the computer is to provide computational capability for maintaining a reference frame and processing sensor measurements to aid the tracking system to follow dynamic targets. Optimal filtering concepts are frequently applied as a part of the processing performed by the computer for the pointing and tracking system.

The pointing and tracking problem is but one example of systems which require the use of reference frames for angular orientation. A historically more prominent example is the inertial navigation problem. Because of its significance, the inertial navigation problem has received considerable study both theoretically and in terms of actual implementations. However, some of the requirements placed on the reference system by the navigation problem are not necessarily consistent with those in a precision pointing and tracking system. Thus, there is some impetus for reviewing the basic foundations of reference system technology.

Generally speaking, there are two methods for maintaining an angular orientation reference for a moving body (an aircraft, a telescope, a gun, for example). Figure 1 is used to illustrate the conceptual difference between the two

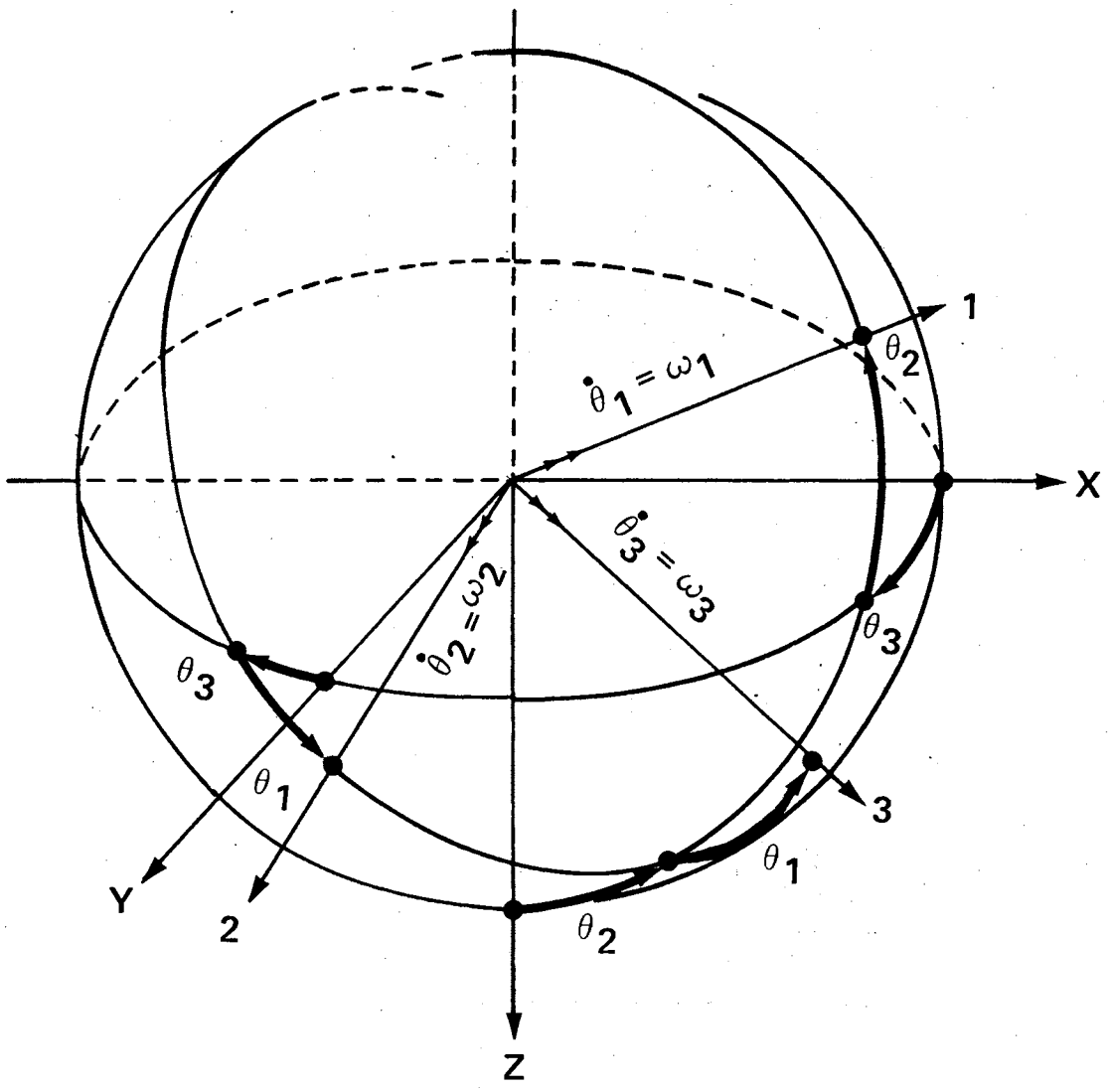


Figure 1. Relation of Inertial and Body-Fixed Coordinate Frames

approaches. It depicts a non-rotating reference frame (denoted xyz) and a moving body (marked 123). The xyz axes do not rotate in inertial space. The 123 axes, on the other hand, are rotating and their rotations may be represented by either the angles θ_1, θ_2, θ_3, or equivalently by a starting position and angular rate history ω_1, ω_2, ω_3. The two types of reference systems are referred to herein as the "stable platform references" and the "strapped-down reference" systems. In the first approach, a gimbal-supported platform is stabilized and kept aligned with the reference axes by using gyros and servo control loops. Angle resolvers are used to sense the orientation of the body (θ_1, θ_2, θ_3) relative to the reference frame. In the second approach, the reference frame orientation is maintained in a computer. This is done by mounting rate sensors directly to the body. The measured rates (ω_1, ω_2, ω_3) are processed by the computer to determine the body angular orientation relative to a reference frame.

The stable platform device (also referred to as an Inertial Measurement Unit (IMU)) has been in use since its development first as navigation and guidance elements for the V-2 rockets in Germany. Considerable developments and improvements have been made in stable platforms (and the associated gyros, servo torquers, and resolvers) over the last twenty-five years. IMU's exist today in many sizes and

levels of performance quality. In many applications, the stable platform concepts must still be considered as the most practical approach for maintaining an angular reference.

The strapped-down reference is a more recent innovation. The major feature (that from which the name is derived) is that the sensors are mounted directly to the rotating body. This provides advantages with respect to size, weight and cost because the gimbals, servoes, resolvers and associated electronics are eliminated. Particular advantage is gained if a digital computer is already a part of the system.

When a strapped-down reference system is chosen over a stable platform system, the designer must then consider problems of computational algorithms and methods for describing angular motion. He has available several alternatives to consider. While it is noted that the application of strapped-down techniques for maintaining reference frames is relatively new (ten to twelve years), the theory of rotational transformations dates back to early mathematics and mechanics researchers such as Euler in 1776, and Hamilton in 1853. Generally, the methods for describing an angular transformation may be divided into three classes which are designated three, four, and nine-parameter methods. (These names come simply from the consideration that the number of variables needed to describe the body reference frame

relative to the fixed frame is three, four and nine respectively.

The major three-parameter method is that due to Euler (13). He introduced the concepts of Euler Angles to represent the rotation of a body to aid his study of astronomy. While the differential equations for the Euler angles could be used to maintain a reference frame in a strapped-down system, this choice is seldom made. There are two reasons: one, the equations have a complicated nonlinear form and two, a singularity exists at certain configurations. The singularity is akin to gimbal lock in a three gimbal system.

There are two four-parameter methods (the fourth parameter is included to eliminate the singularity). Again, the basic formulations date back to historically prominent investigators. The four-parameter methods for describing an angular rotation between two axes sets are the quaternion and the Cayley-Klein transformations. The major investigation of the quaternion is due to Hamilton (20), (21). Gauss, Rodrigues, and Cayley also did work in this area. The so-called Cayley-Klein four-parameter method was presented by Klein (24). Goldstein (14) gives a good theoretical review of the Cayley-Klein transformation. A summary in the application of the quaternion technique for implementing a strapped-down reference system is presented by Wilcox (37).

The nine-parameter method is based on the concept of direction cosines. (The nine parameters are the cosines of the nine angles that relate the three body axes to the reference frame.) This method, too, has its foundations in classical mechanics. The method is attractive in the sense that the equations needed to propagate the parameters as a body moves are linear differential equations. Wiener (36), in one of the early contributions to the development of strapped-down reference systems, utilized the nine-parameter method. His system included a digital differential analyzer (DDA) to propagate the solutions to the direction cosine equations and thus maintain the reference frame. (The DDA was a special purpose digital computer implemented to solve differential equations.) The nine-parameter technique may not be the most efficient for use on a general purpose machine.

The propagation of the transformations relating angular orientation between a reference frame and a rotating body involves the common problem of solving a set of differential equations in the reference system computer. A large number of different algorithms have been proposed for use in reference system computers. Factors such as accuracy, sensor characteristics, computer performance characteristics, and

designer/programmer preferences generally determine which method is implemented.

The methods for numerically solving the strapped-down reference system equations may be broadly divided into two classes: one, methods based on the assumption of commutativity; and two, methods which do not depend on the assumption. Commutativity implies that the final orientation of a body is independent of the order in which the intermediate rotations are performed. For small rotations the difference in final orientations obtained by doing the rotations in two different sequences will be small. All currently used propagation procedures are based on the commutativity approximation (that there is no difference in final orientation) except the one originated in this report.

The new method which is developed and evaluated in later sections of this report has as its basis techniques which are not new. To the author's knowledge, however, no previous attempt has been made to apply variational-based approaches to the problem of updating direction cosines in a strapped-down reference system. The method has been classed as a variational method because it was first derived by the methods of variational calculus. There are alternative derivations, however, and the approximation is more suitably called Galerkin's technique.

The historical background for the so-called variational methods dates back to Rayleigh (30) who was interested in solving vibration problems. His problem was in the form of a variational calculus problem and involved the extremization of a functional over an infinite-dimensional function space. Rayleigh's approximation method converted the variational calculus problem into one of ordinary calculus by approximating the true solution by doing the extremization over a finite-dimensional subspace. Rayleigh's method was extended by Ritz (31), still retaining the variational calculus formulation as a foundation. The extended method now bears the name Rayleigh-Ritz method. The Rayleigh-Ritz method is constrained in applicability to those physical systems for which a Lagrangian (kinetic energy minus potential energy) can be defined. Galerkin (14) extended the variational concept to cases in which the motion equations are known but for which the energy formulation may not be known or may not have meaning. (Such is the case in the problem of propagating direction cosines.) Kantorovich and Krylov (23) provides an excellent classical treatment of the Ritz-Galerkin (as it is frequently referred to) approximation method and an introduction to variational problems. Atkinson (1) and Kantorovich and Akilov (22) provide discussions of the Galerkin approach in the framework of modern functional analysis.

Traditionally, the variational approaches were concerned primarily with finding "closed-form" approximate solutions as opposed to numerical solutions. In the reference system computer, of course, the solutions are numerical in nature. Ebbesen (12) showed that the variational idea could successfully be used to propagate solutions to differential equations involving high frequency parasitic effects. The results of Ebbesen's work provided motivation that a variational based solution technique might be advantageous for strapped-down reference frame propagation.

2. Research Objectives

The overall objective of this work is to develop and investigate techniques that can be used to improve the performance capabilities of high accuracy pointing and tracking systems. The work is concerned primarily with the problem of maintaining a reference frame in a computer so that dynamic target motions can be estimated.

More specific intermediate objectives include:

1. Development of a computational algorithm based on the variational concept of Galerkin suitable for use in propagating the direction cosines of the strapped-down reference system. This includes a review of the conventional approaches.

2. Perform comparisons of the new algorithm with existing strapped-down propagation methods. Comparisons are made on the basis of accuracy and required computational time.
3. Evaluate the new algorithm for application in high accuracy pointing and tracking systems. A simulation model of the AFWL's airborne pointing and tracking system is used for this purpose.

3. Plan of Presentation

Section 1 of Chapter II gives some background in rotational transformations. This is necessary to provide common ground for later discussions. In this section Euler angles will be introduced and the non-commutativity of angular transformations discussed. Section 2 of Chapter II outlines the commutative algorithms. Theoretical justification for the use of these algorithms is given. The update equations are given for both the nine-parameter method and for two four-parameter methods. Orthogonality correction routines are discussed and their applicability considered. Section 3 of Chapter II gives some theoretical background for Galerkin's technique and outlines the algorithm using this technique. Chapter III details the methods used for comparing the algorithms and shows results of these comparisons. Chapter IV

considers aspects of the problem which apply directly to the pointing and tracking system and gives some simulation results for that system. Chapter V contains conclusions and recommendations. Appendix A gives proof which shows that the Galerkin technique reduces in the limit to an Euler method. Appendix B details a modal matrix technique which decouples the cosine equations and results in a single scalar integro-differential equation for each element of the cosine matrix. Appendix C shows the connection between the Rayleigh-Ritz method and Galerkin's method. Appendix D gives a general convergence proof for the Galerkin technique and for the commutative technique.

CHAPTER II

THEORETICAL CONSIDERATIONS

1. Angular Transformations

The angular relation between two axis sets is uniquely determined by three numbers called Euler angles. The Euler angles for a body-fixed axis set may be determined in the following manner. Starting with the body axes (labeled X_B, Y_B, Z_B) and the inertial axes (labeled X_I, Y_I, Z_I) aligned carry out the following set of rotations:

1. Rotate the body about its Z_B axis by θ_Z degrees.
2. Rotate the body about its Y_B axis by θ_Y degrees.
3. Rotate the body about its X_B axis by θ_X degrees.

The resulting orientation of the body axis set is completely determined by the Euler angles (θ_X, θ_Y, θ_Z). Let C be a matrix of direction cosines which relates the body-fixed axis set to the inertial axis set. Any vector $\underline{X}_B$ measured in the body-fixed coordinate frame may be transformed to $\underline{X}_I$ which is the same vector measured in the inertial frame. The transformation may be accomplished by:

$$\underline{X}_I = C\underline{X}_B \tag{2.1}$$

where

$$\underline{X}_I = \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix} \quad \text{and} \quad \underline{X}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} \quad .$$

If a matrix C_1 relates the X_B, Y_B, Z_B axis frame to some intermediate frame, and C_2 relates the intermediate frame to the inertial frame the total transformation may be accomplished by:

$$\underline{X}_I = C_2 C_1 \underline{X}_B \quad . \tag{2.2}$$

The transformations represented by C_2 and C_1 are not commutative, in general, however. That is, in general,

$$C_1 C_2 \neq C_2 C_1 \quad .$$

Figure 2 illustrates the non-commutativity of successive rotations. In that figure, the rotations θ_X and θ_Y are the same in both cases, but are carried out in different orders. The resulting orientations are not the same.

Let C(a,b,c) denote the direction cosine matrix for a set of Euler angles, c, b, a, carried out in that order, where c corresponds to θ_Z, b to θ_Y, and a to θ_X. Since the rotations c, b and a are carried out separately, then by Equation (2.2) one obtains

$$C(a,b,c) = C(a,o,o)\ C(o,b,o)\ C(o,o,c) \quad . \tag{2.3}$$

Let

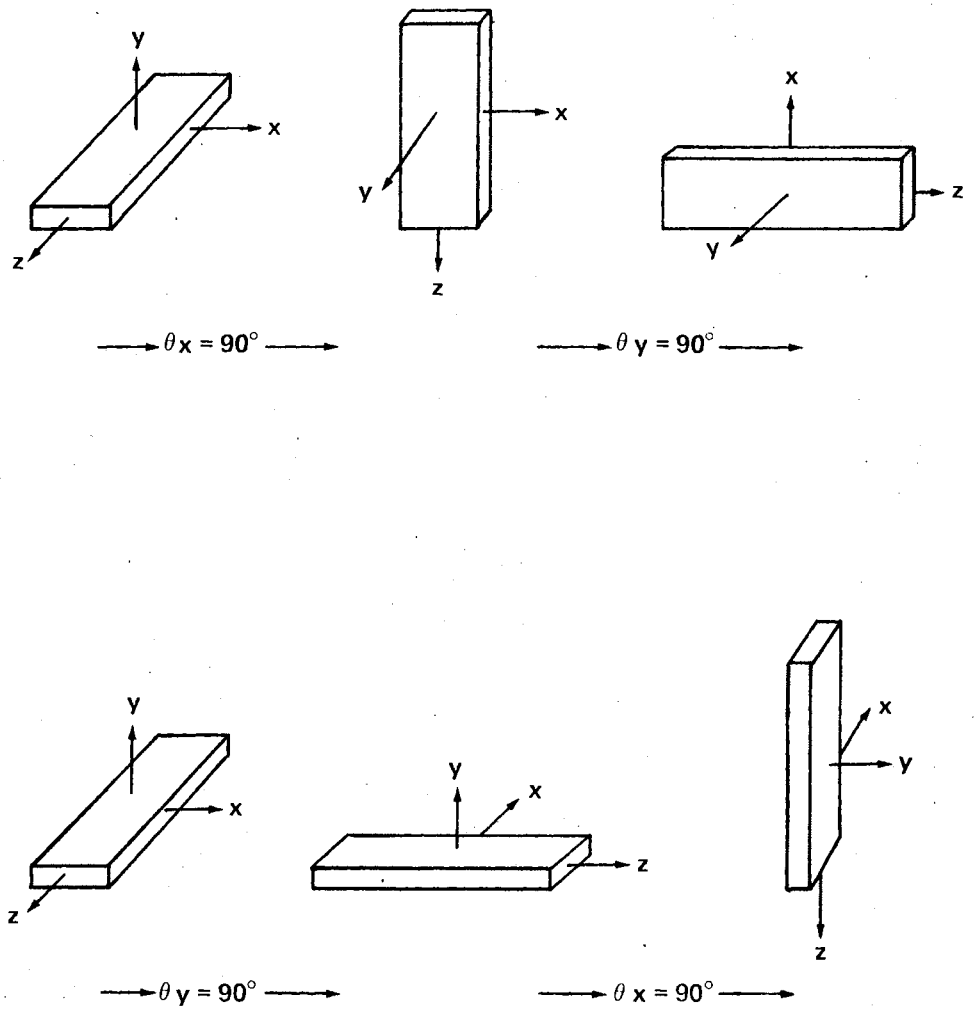


Figure 2. Illustration of the Non-Commutativity of Rotations

$$C1 \triangleq \cos a, \qquad S1 \triangleq \sin a,$$

$$C2 \triangleq \cos b, \qquad S2 \triangleq \sin b,$$

$$C3 \triangleq \cos c, \text{ and } S3 \triangleq \sin c.$$

From the geometry of the problem

$$C(o,o,c) = \begin{bmatrix} C1 & S1 & 0 \\ -S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C(o,b,o) = \begin{bmatrix} C2 & 0 & -S2 \\ 0 & 1 & 0 \\ S2 & 0 & -C2 \end{bmatrix}$$

and

$$C(a,o,o) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C3 & S3 \\ 0 & -S3 & C3 \end{bmatrix} .$$

By multiplying the three matrices together, one finds that the direction cosine for the total rotation (a,b,c) can be written as:

$$C(a,b,c) = \begin{bmatrix} C1C2 & S1C3-C1S2S3 & S1S2-C1S2C3 \\ -S1C2 & C1C3-S1S2S3 & C1S3+S1S2C3 \\ S2 & -C2S3 & C2C3 \end{bmatrix} . \quad (2.4)$$

Now consider the case where the axis set is rotated infinitesimally. Let an infinitesimal rotation be denoted by $d\theta$. Then expansion of the elements of the cosine matrix in a Taylor series and neglecting higher order terms gives

$$C(o,o,d\theta_z) = \begin{bmatrix} 1 & d\theta_z & 0 \\ -d\theta_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I + \begin{bmatrix} 0 & d\theta_z & 0 \\ -d\theta_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

and

$$C(d\theta_x,d\theta_y,d\theta_z) = C(o,o,d\theta_z)\ C(o,\theta_y,o)\ C(\theta_x,o,o) \quad . \tag{2.5}$$

By multiplying these out and neglecting higher order terms, we arrive at

$$C(d\theta_x,d\theta_y,d\theta_z) = I + \begin{bmatrix} 0 & d\theta_z & -d\theta_y \\ -d\theta_z & 0 & d\theta_x \\ d\theta_y & -d\theta_x & 0 \end{bmatrix} . \tag{2.6}$$

A vector X which is transformed to X' by this infinitesimal rotation will be given by

$$X' = (I + d\theta)\ X \tag{2.7}$$

where

$$d\theta = \begin{bmatrix} 0 & d\theta_z & -d\theta_y \\ -d\theta_z & 0 & d\theta_x \\ d\theta_y & -d\theta_x & 0 \end{bmatrix} . \tag{2.8}$$

Alternatively,

$$X' - X = d\theta X$$

and

$$dX = d\Theta X \quad .$$

If Θ is a function of time, then

$$\frac{dX}{dt} = \frac{d\Theta}{dt} X \quad . \tag{2.9}$$

The columns of the direction cosine matrix are vectors. Each is a unit vector aligned with one of the body-fixed axes. Denoting the first column by C_1, we arrive at the relation:

$$\frac{dC_1}{dt} = \frac{d\Theta C_1}{dt} \quad . \tag{2.10}$$

Since the columns of the direction cosine matrix are independent, a matrix equation of the form

$$\frac{dC}{dt} = \frac{d\Theta}{dt} C \tag{2.11}$$

may be written, where C is a 3 x 3 matrix. This is the time-varying linear differential equation which must be solved to propagate the direction cosines.

2. Commutative Approximations

2.1 Theoretical Justification

First, we shall give some theoretical justifications for the use of a commutative approximation. This analysis is due to Pipes (29). Given a differential equation

$$\dot{x} = A(t)x \quad , \tag{2.12}$$

define $\bar{A}$ by

$$\bar{A} \triangleq \frac{1}{b-a} \int_a^b A(s)ds \tag{2.13}$$

and

$$e(t) \triangleq A(t) - \bar{A} \quad . \tag{2.14}$$

Then $\dot{x}$ may be written as

$$\dot{x} = \bar{A}x + e(t)x \quad . \tag{2.15}$$

Furthermore, the state transition matrix $\Phi(t,a), \forall\, t \in [a,b]$, must satisfy:

$$\frac{d\ \Phi(t,a)}{dt} = A(t)\ \Phi(t,a)$$

$$= \bar{A}\ \Phi(t,a) + e(t)\ \Phi(t,a) \quad . \tag{2.16}$$

An exact expression for the state transition matrix is then given by the standard variation of parameters formula,

$$\Phi(b,a) = \exp[\bar{A}(b-a)] \cdot [I + \int_b^a \exp(-\bar{A}\tau)\, e(\tau)\, \Phi(\tau,a)\, d\tau] \quad . \tag{2.17}$$

The commutativity assumption is then that

$$e(t) \cong 0 \qquad \forall\, t \in [b,a] \quad .$$

With this assumption the expression for $\Phi(b,a)$ becomes

$$\Phi(b,a) \cong \exp[\bar{A}(b-a)] \quad . \tag{2.18}$$

The validity of the assumption depends on how fast A(t) changes relative to the time step (b-a). If A(t) varies

slowly, then a large time step may be used to propagate the solution with acceptable accuracy. The approximation becomes exact in the limit, i.e.,

$$\lim_{b \to a} \exp[\bar{A}(b-a)] = I \quad .$$

The solution may be further approximated by expanding the exponential in a Taylor series and truncating after a few terms:

$$\exp[\bar{A}(b-a)] = I + \bar{A}(b-a) + \frac{\bar{A}^2(b-a)^2}{2} + \frac{\bar{A}^3(b-a)^3}{3!} + \ldots$$

$$\cong I + \bar{A}(b-a) \quad . \tag{2.19}$$

If the series is truncated to a very few terms, then the time step may have to be further reduced in order to reduce the significance of the higher-order terms.

One further approximation which has been used in some applications is the following:

$$B \triangleq A(a)$$

$$f(t) \triangleq A(t) - A(a) \quad . \tag{2.20}$$

Now, $\Phi(t,a)$ $\forall t \in [a,b]$, must satisfy

$$\frac{d\,\Phi(t,a)}{dt} = B\,\Phi(t,a) + f(t)\,\Phi(t,a) \tag{2.21}$$

and $\Phi(b,a)$ may be written as:

$$\Phi(b,a) = \exp[B(b-a)] \cdot \left[I - \int_a^b \exp(-B\tau)\, f(\tau)\, \Phi(\tau,a)\, d\tau\right] . \tag{2.22}$$

The approximation is now based on

$$f(t) \cong 0 \quad .$$

It can be shown that f(t) is further from zero than e(t). One measure, for example, yields

$$\int_a^b (e(s))^2 \, ds \leq \int_a^b (f(s))^2 \, ds \quad . \tag{2.23}$$

Therefore, this approximation is not as good as the first one presented. The situation may in practice become quite bad for the second approximation if the measurement of A(a) is noisy, and a truncated series is used for the exponential. The values of the B matrix can be large enough to force the use of a much smaller time step than the first approximation requires. It should be noted that a two term expansion of the second approximation amounts to an Euler method integration of the differential equation.

2.2 Nine-Parameter Solution

We desire to find the angular relation between two axis sets. One way to do this is by solving Equation (2.11) which was given at the end of Section 2.1. In that equation

$$\frac{d\theta}{dt} = \begin{bmatrix} 0 & \frac{d\theta_3}{dt} & -\frac{d\theta_2}{dt} \\ -\frac{d\theta_3}{dt} & 0 & \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} & -\frac{d\theta_1}{dt} & 0 \end{bmatrix} .$$

Define

$$\omega_3(t) \triangleq \frac{d\theta_3}{dt} , \quad \omega_2(t) \triangleq \frac{d\theta_2}{dt} , \quad \omega_1(t) \triangleq \frac{d\theta_1}{dt} .$$

Then

$$\frac{d\theta}{dt} = \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) \\ -\omega_3(t) & 0 & \omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{bmatrix} . \qquad (2.24)$$

Define

$$\frac{d\theta}{dt} \triangleq (t) .$$

This gives

$$\frac{dC}{dt} = (t)C .$$

In the case where (t) is a constant, i.e.,

(t) =

(not a function of time) the equation has an analytical solution:

$$C(t) = \exp[\ t]\ C(o). \qquad (2.25)$$

Several methods are available to find the matrix function exp[t]. One such method is Sylvester's expansion theorem (see Dorf (8)).

The state transition matrix for constant is:

$$\exp[\ t] = \frac{1}{\omega_1^2+\omega_2^2+\omega_3^2}\begin{bmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_3\omega_2 \\ \omega_1\omega_3 & \omega_3\omega_2 & \omega_3^2 \end{bmatrix}$$

$$- \frac{\cos\left(\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}\,t\right)}{\omega_1^2+\omega_2^2+\omega_3^2}\begin{bmatrix} -(\omega_3^2+\omega_2^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_3^2) & \omega_3\omega_2 \\ \omega_1\omega_3 & \omega_3\omega_2 & -(\omega_1^2+\omega_2^2) \end{bmatrix}$$

$$+ \frac{\sin\left(\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}\,t\right)}{(\omega_1^2+\omega_2^2+\omega_3^2)^{1/2}}\begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} . \qquad (2.26)$$

2.3 Quaternions

Hamilton defines a Quaternion to be the quotient of two vectors. In general, a quaternion is a complex quantity of the form

$$\rho = \rho_1 i + \rho_2 j + \rho_3 k + \rho_4 \ . \qquad (2.27)$$

The magnitude of ρ is denoted by ρ_o and can be expressed as

$$\rho_o = \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 \ . \qquad (2.28)$$

Let a general three vector be given by

$$X = X_1 i + X_2 j + X_3 k \ . \qquad (2.29)$$

Then the orthogonal rotation of X_B into X_I may be written in the form

$$X_I = \rho X_B \rho^* \quad . \tag{2.30}$$

The (*) indicates complex conjugation and the complex quantities i, j, k obey the following laws of multiplication:

$$i^2 = j^2 = k^2 = -1$$

$$ij = k; \quad ji = -k$$

$$jk = i; \quad kj = -i$$

$$ki = j; \quad ik = -j \quad .$$

By carrying out the operations indicated in Equation (2.30), one can show that the cosine matrix can be related to the quaternion ρ, i.e.,

$$C = \begin{bmatrix} \rho_1^2-\rho_2^2-\rho_3^2+\rho_4^2 & 2(\rho_1\rho_2-\rho_3\rho_4) & 2(\rho_3\rho_1+\rho_2\rho_4) \\ 2(\rho_1\rho_2+\rho_3\rho_4) & -\rho_1^2+\rho_2^2-\rho_3^2+\rho_4^2 & 2(\rho_2\rho_3+\rho_1\rho_4) \\ 2(\rho_3\rho_1-\rho_2\rho_4) & 2(\rho_2\rho_3+\rho_1\rho_4) & -\rho_1^2-\rho_2^2+\rho_3^2+\rho_4^2 \end{bmatrix} . \tag{2.31}$$

The quaternion at any time t may be found by solving

$$\dot{\rho}(t) = \rho(t)\frac{q(t)}{2} \tag{2.32}$$

where

$$q(t) = \omega_1 i + \omega_2 j + \omega_3 k + 0 \quad . \tag{2.33}$$

Define

$$\theta_i \triangleq \int_a^b \omega_i(t)\, dt, \qquad i = 1, 2, 3; \tag{2.34}$$

and

$$\theta \triangleq \theta_1 i + \theta_2 j + \theta_3 k + 0 \quad . \tag{2.35}$$

Then with the commutativity assumption, an approximate solution of the form

$$\rho(b) \cong \exp\left[\frac{\theta}{2}\right] \tag{2.36}$$

may be written. Define

$$\theta_o \triangleq \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2} \quad .$$

Then

$$\exp\left[\frac{\theta}{2}\right] = (\theta_1 i + \theta_2 j + \theta_3 k) \frac{\sin(\theta_o/2)}{\theta_o} + \cos\frac{\theta_o}{2} \quad . \tag{2.37}$$

The quaternion ρ may be updated by using

$$\rho(NT + T) = \rho(NT) \exp\left[\frac{\theta}{2}\right] \quad . \tag{2.38}$$

2.4 Cayley-Klein

Consider a two-dimensional space with complex axes U and V. A linear transformation in this space may be defined

$$U' = \alpha u + \beta v$$

$$V' = \sigma u + \delta v$$

or

$$\begin{bmatrix} U' \\ V' \end{bmatrix} = Q \begin{bmatrix} U \\ V \end{bmatrix} \tag{2.39}$$

where

$$Q = \begin{bmatrix} \alpha & \beta \\ \sigma & \delta \end{bmatrix} . \tag{2.40}$$

The parameters α, β, σ and δ are the four parameters of the Cayley-Klein method. Impose the restriction that Q is unitary and that its determinant is +1:

$$Q^{\dagger} Q = 1 = QQ^{\dagger} \qquad (\dagger \text{ indicates transposed conjugate}) \tag{2.41}$$

and

$$\alpha\delta - \beta\sigma = +1 .$$

The unitary requirement makes the matrix be of the form

$$Q = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix} . \tag{2.42}$$

Let P be a matrix operator in the UV plane of the form

$$P = \begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix} \tag{2.43}$$

where x, y and z are real numbers. Since Q is unitary

$$Q^{\dagger} = Q^{-1}$$

and

$$P' = QPQ^{\dagger} \tag{2.44}$$

is a similarity transformation.

The matrix P is self-adjoint or hermitean. The determinant is invariant under a similarity transformation or

$$P' = -(x^2+y^2+z^2) = -(x'^2+y'^2+z'^2) = P' . \tag{2.45}$$

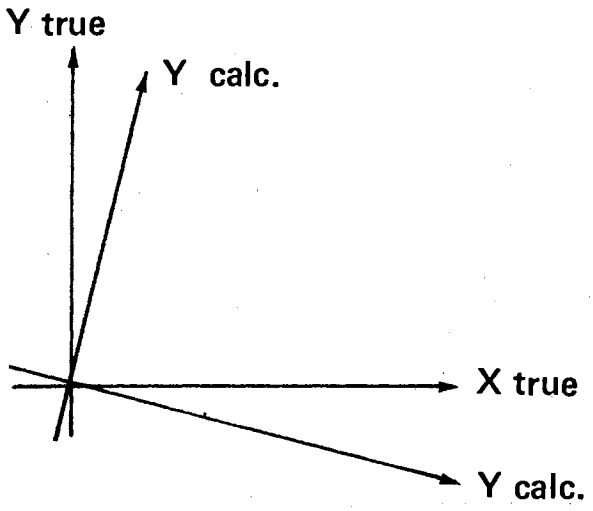
Equation (2.45) is the orthogonality condition. If x, y and z are the coordinates of a point, the distance from the point of the origin does not change after a rotation of the coordinate frame about its origin. The similarity transformation in the UV plane can be used to represent the coordinate transformation in the cartesian plane. The Cayley-Klein parameters are related to the quaternion parameters by:

$$\begin{aligned} \alpha &= \rho_4 - i\rho_3 \\ \beta &= -\rho_2 - i\rho_1 \\ \sigma &= \rho_2 - i\rho_1 \\ \delta &= \rho_4 - \rho_3 \end{aligned} \qquad (2.46)$$

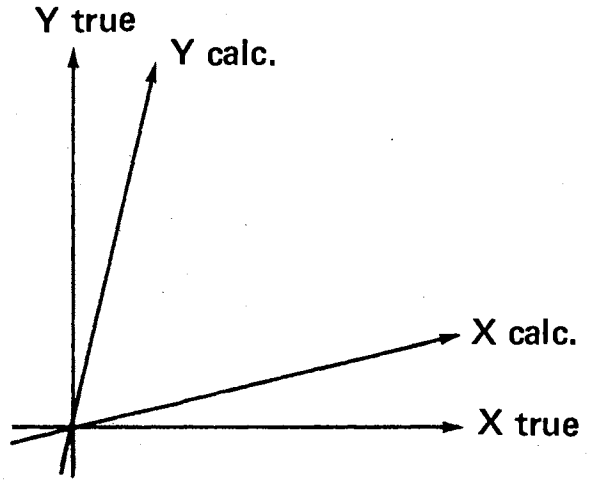
The differential equation for Q and the solution follow directly from those for the quaternions.

2.5 Orthogonality Corrections

The direction cosines relating the two axis frames should remain orthogonal; that is, the projection of any axis on any other should be zero. Mortensen (27) gives the two dimensional picture to show what is happening. There are two types of errors that may occur in calculating the orientation of the axis sets, skew error and orthogonality error. Figure 3a shows skew error. Skew error rotates the axis set away from its true position. Figure 3b shows orthogonality error. In the case of orthogonality error, the axes



a.) Skew Error



b.) Orthogonality Error

Figure 3. Examples of Skew Error and Orthogonality Error

are no longer at right angles to each other. The purpose of an orthogonality correction is to remove this second type of error. That is to insure that the calculated axes are at right angles to each other. The Gram-Schmidt process is commonly used to correct the direction cosine matrix for non-orthogonality. It proceeds as follows. Consider the three columns of the direction cosine matrix C to be C_1, C_2 and C_3. The projection of any one onto another is just their dot product. Begin with C_1 and successively remove the components of C_2 and C_3.

Normalize C_1 by:

$$\overline{C_1} = \frac{C_1}{C_1} \tag{2.47}$$

where C_1 indicates the magnitude of C_1. Then find projection of C_2 onto $\overline{C_1}$ by

$$C_{2 \text{ on } 1} = (\overline{C_1}, C_2) \tag{2.48}$$

where

$$(x,y) = x_1y_1 + x_2y_2 + x_3y_3 \tag{2.49}$$

The corrected C_2 is obtained by subtracting this projection and normalizing.

$$\overline{C_2} = \frac{C_2-(\overline{C_1},\ C_2)\ \overline{C_1}}{C_2-(\overline{C_1},\ C_2)\ \overline{C_1}} \tag{2.50}$$

and finally find the projections of C_3 onto $\overline{C_1}$, C_3 onto $\overline{C_2}$, subtract them and normalize C_3.

$$\overline{C_3} = \frac{C_3 - (\overline{C_1},\ C_3)\ \overline{C_1} - (\overline{C_2},\ C_3)\ \overline{C_2}}{\left| C_3 - (\overline{C_1},\ C_3)\ \overline{C_1} - (\overline{C_2},\ C_3)\ \overline{C_2} \right|} \tag{2.51}$$

This series of operations renders the resulting vectors $\overline{C_1}$, $\overline{C_2}$ and $\overline{C_3}$ orthonormal, but may introduce some error. The trouble is that the first vector is always assumed to be correct. This amounts to correcting the error in Figure 4a by going to Figure 4b. This certainly removes the orthogonality error, but converts it to skew error.

The orthogonality condition for the quaternion update equation is that

$$\sqrt{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2} = 1 \quad . \tag{2.52}$$

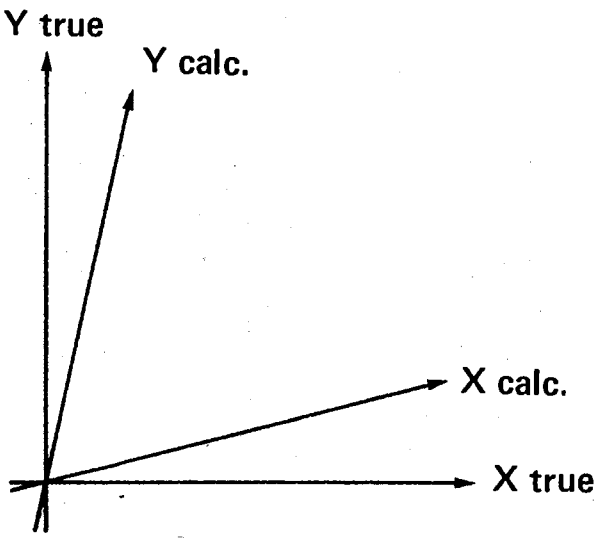
Define

$$|\rho| \triangleq \sqrt{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2} \tag{2.53}$$

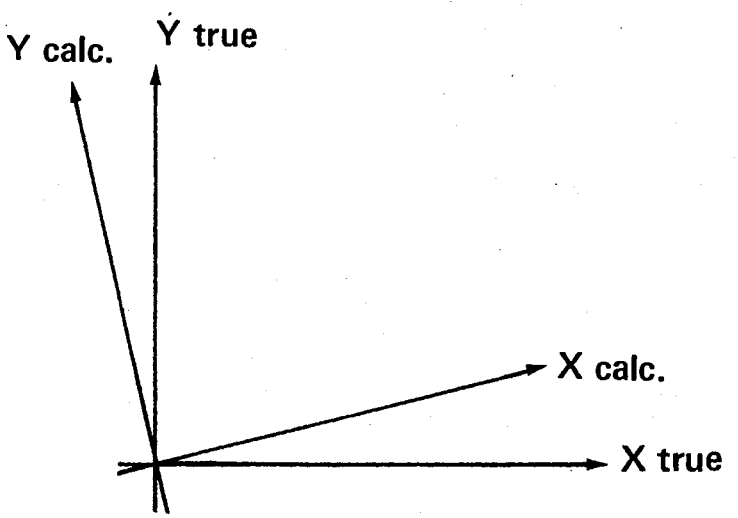
then the orthogonality correction is

$$\overline{\rho_i} = \frac{\rho_i}{|\rho_i|} \quad . \tag{2.54}$$

This orthogonality correction does not assume that any one of the four parameters is more accurate than the others. For this reason the orthogonality correction which is applied to the quaternion is generally a more accurate correction.



a.) Orthogonality Error



b.) Gram-Schmidt Correction

Figure 4. Gram-Schmidt Orthogonality Correction

3. Analog Computer Solution

Although the digital computer solution of the cosine equation is somewhat involved (as is seen in the algorithms presented here), the analog solution is quite straightforward. Consider the first column of the direction cosine equation

$$\begin{bmatrix} \dot{C}_1 \\ \dot{C}_2 \\ \dot{C}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) \\ -\omega_3(t) & 0 & \omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

This vector matrix equation may be written as three first-order coupled equations:

$$\begin{aligned} \dot{C}_1 &= \omega_3(t)\, C_2 - \omega_2(t)\, C_3 \\ \dot{C}_2 &= -\omega_3(t)\, C_1 + \omega_1(t)\, C_3 \\ \dot{C}_3 &= \omega_2(t)\, C_1 - \omega_1(t)\, C_2 \end{aligned} \tag{2.55}$$

A block diagram of the analog computer setup necessary to solve these equations is shown in Figure 5.

The EAI TR-48 at OSU was used to solve these equations for a particular time history of the omega matirx. The results of the analog computer solution are compared with the results of a digital computer solution in Figures 6a and 6b . Those figures show time histories of particular elements of the cosine matrix as calculated by each technique. The agreement is not as good as might be desired. Part of

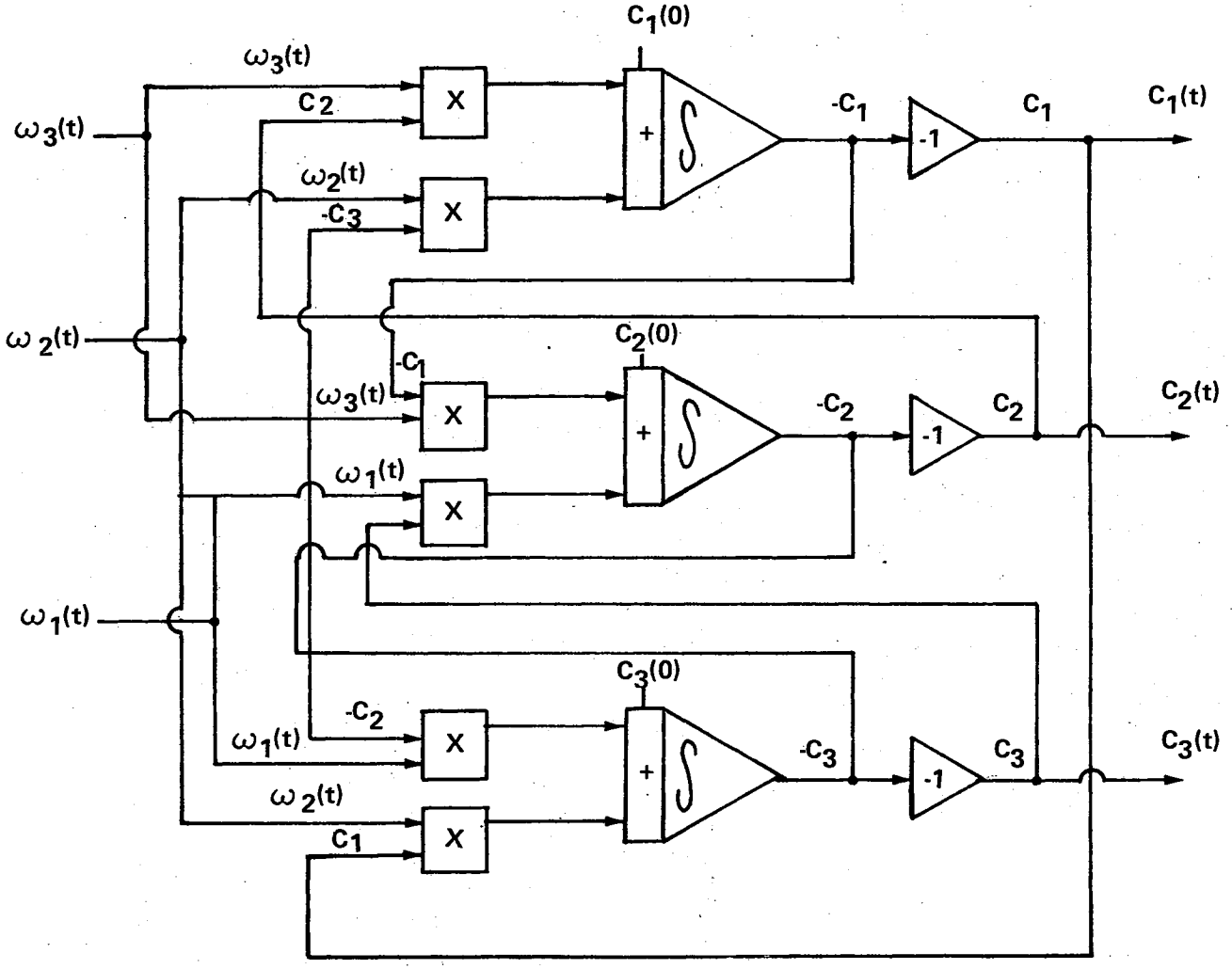
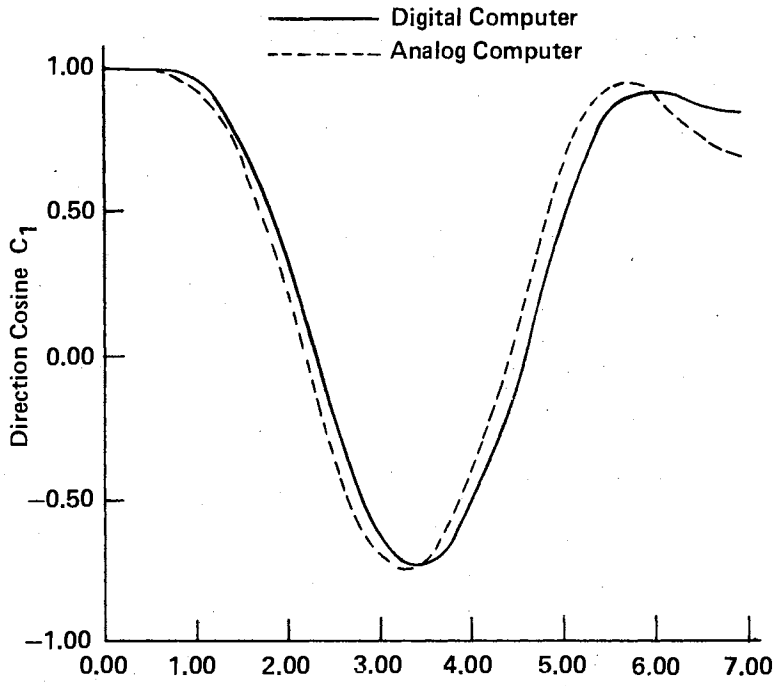
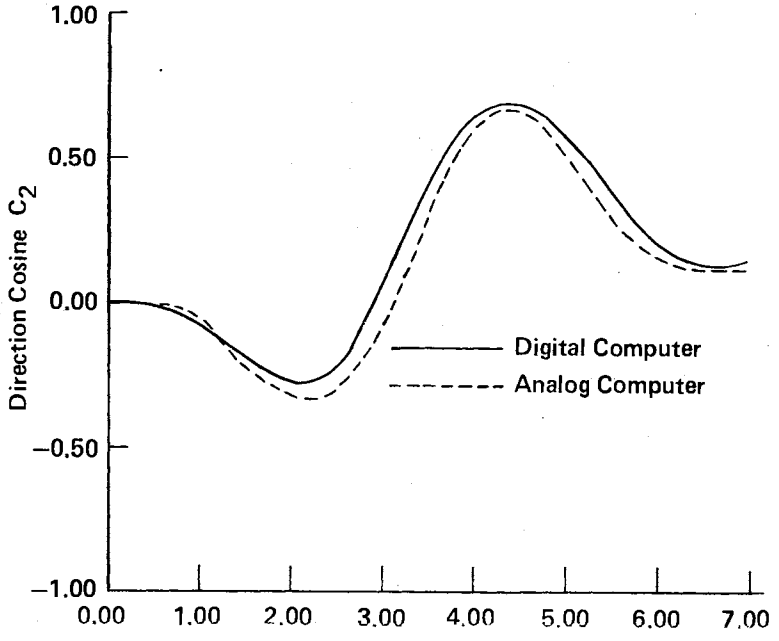


Figure 5. Analog Computer Block Diagram



a.) Time (seconds) C_1



b.) Time (seconds) C_2

Figure 6. Comparisons of Analog Computer Results With Digital Computer Results

of the problem is the inaccuracy of the multipliers that the analog machine used. The solution would be better with more accurate equipment. There is, however, a limit of about 1% to this accuracy. As seen in Figure 5, each vector in the cosine equation would require three integrators, three inverters, and six multipliers. If high gain amps are used to provide the integration and inversion functions, then the total requirements for the system are eighteen high gain amplifiers and eighteen multipliers.

4. Galerkin Technique

The application of Galerkin's technique to the general time varying linear system is considered first. The result is then specialized to the direction cosine equation. A convenient solution is not available for the general time varying linear equation as it is in the time invariant case. This forces us to use some approximation technique to get close to a true solution. One class of approximations stems from assuming commutativity; some of these have already been presented. This assumption can lead to a build-up of error. Galerkin's technique may also be applied to the time-varying linear system to yield an approximate solution. The use of Galerkin's technique does not depend on the piecewise time-invariance assumption. It depends instead on the assumption

that the solution may be represented by a finite power series, finite trigonometric series, or some arbitrary series of functions.

Given a vector-matrix differential equation of the form:

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) \tag{2.56}$$

where $\dot{x}(t)$, $x(t)$ and $B(t)$ are k x 1 vector valued functions of time; $A(t)$ is a k x k matrix valued function of time, and $u(t)$ is a scalar function; and given an initial condition vector for the x vector:

$$x(t_o) = x_o \tag{2.57}$$

we desire to find the vector function $x(t)$ which solves Equation (2.56). Require that matrix-valued function $A(t)$ and the product $B(t) \cdot u(t)$ are continuous except possibly at a finite number of points and have finite right-hand and left-hand limits at those discontinuities. For this case, the time-varying linear system in Equation (2.56) satisfies the Lipschitz condition according to Desoer (9). This insures that a solution $x(t)$ exists and is unique. However, to find the solution is non-trivial. A general solution to Equation (2.56) cannot be found in convenient form.

A closed form solution does exist for the time-varying system if the following is true.

$$A(t_1)\, A(t_2) = A(t_2)\, A(t_1) \quad . \tag{2.58}$$

This commutativity condition is very restrictive, but allows a closed form solution, i.e.:

$$x(t) = \exp\left(\int_{t_o}^{t} A(s)\,ds\right)\left[x_o + \int_{t_o}^{t} \exp\left(\int_{t_o}^{s} A(\tau)\,d\tau\right) B(s)\,u(s)\,ds\right] . \tag{2.59}$$

This is the familiar variation of parameters solution. The commutativity condition is seldom satisfied, however.

There are some interesting cases where Equation (2.59) may be applied. One of these occurs when

$$A(t) = C\,f(t) \tag{2.60}$$

where C is a k x k constant matrix and f(t) is a scalar function. Another is the case where the A matrix is piecewise constant, i.e.,

$$\begin{aligned} A(t) &= A_1 && 0 \le t < t_1 \\ &= A_2 && t_1 \le t < t_2 \\ &\;\;\vdots && \;\;\vdots \end{aligned} \tag{2.61}$$

where A_1, A_2, ... are constant matrices. For this case, Equation (2.59) may be applied recursively to obtain an exact solution.

For those cases where condition (2.58) is not met, we must be satisfied with some approximation to the true

solution. At this point most investigators turn to the digital computer and numerical integration techniques. Application of a numerical integration routine for a commutative solution amounts to assuming that condition (2.61) holds. An improper commutativity assumption can lead to error build-up.

Another method for solving Equation (2.56) is Galerkin's technique. This method does not assume commutativity and is straightforward to apply. In order to apply Galerkin's technique, first integrate both sides of Equation (2.56).

$$x(t) = \int_{t_o}^{t} A(s)x(s)ds + \int_{t_o}^{t} B(s)u(s)ds + x_o \quad . \quad (2.62)$$

Equation (2.62) is a special case of the Volterra Integral Equation. Rearrange the integral equation

$$x(t) - \int_{t_o}^{t} A(s)x(s)ds = \int_{t_o}^{t} B(s)u(s)ds + x_o \quad . \quad (2.63)$$

Define a linear matrix operator L by:

$$L\,x(t) \triangleq x(t) - \int_{t_o}^{t} A(s)x(s)\,ds \quad . \quad (2.64)$$

Define a vector-valued function f(t) by:

$$f(t) \triangleq \int_{t_o}^{t} B(s)u(s)\,ds + x_o \quad . \quad (2.65)$$

The Equation (2.63) may be rewritten as:

$$L\,x(t) = f(t) \quad . \quad (2.66)$$

This change is made solely to simplify the equation. The integral equation defined by (2.66) is called a Volterra equation. The properties of the Volterra operator have been studied and results are available in standard texts such as Lovitt (26) or Petrovskii (28). The inverse of the Volterra operator exists (for suitable constraints on A(t) and B(t)·u(t)). This insures again that x(t) exists and is unique.

To apply Galerkin's technique, approximate x(t) by $x_n(t)$ where:

$$x_n(t) = \sum_{i=0}^{n} a_i \varphi_i(t) \quad . \tag{2.67}$$

The $\varphi_i(t)$ are scalar functions of time and the a_i are k x 1 vectors which are to be determined. The $\varphi_i(t)$ might be orthogonal polynomials or a particular set of functions if something is known about the form of x(t). Define the dot product of two real-valued functions by:

$$(g,h)_t \triangleq \int_{t_o}^{t} g(s)\, h(s)\, ds \quad , \tag{2.68}$$

where g(t) and h(t) may be scalar or vector so long as they obey the rules of matrix multiplication, and standard rules for a dot product:

$$(g,g)_t \geq 0$$

and

$$(g,g)_t = 0 \text{ , if and only if } g \equiv 0 \text{ .}$$

Then determine the vectors a_i by solving

$$(Lx_n(t), \varphi_j(t)) = (f(t), \varphi_j(t)) \tag{2.69}$$

for each $j = 0, 1, \ldots, n$. Here $Lx_n(x)$ is a column vector valued function and $\varphi_j(t)$ is scalar. The notation $(Lx_n(t), \varphi_j(t))$ is used then to denote a vector, each entry of which is the dot product of the corresponding element of $Lx_n(t)$ and the scalar function $\varphi_j(t)$. Substitution of (2.67) and use of the linearity of L and of the dot product gives

$$\sum_{i=1}^{n} (L\varphi_i(t), \varphi_j(t))\ a_i = (f(t), \varphi_j(t)) \tag{2.70}$$

for each $j = 0, 1, \ldots, n$. Here $L\varphi_i(t)$ is a matrix valued function. The notation $(L\varphi_i(t), \varphi_j(t))$ is used to denote a matrix, each entry of which is the dot product of the corresponding element of $L\varphi_i(t)$ and the scalar function $\varphi_j(t)$. This gives a system of linear equations to solve, i.e.,

$$\begin{bmatrix} & & \\ & (L\varphi_i(t),\varphi_j(t) & \\ & & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} (f(t), \varphi_1(t)) \\ (f(t), \varphi_2(t)) \\ \vdots \\ (f(t), \varphi_n(t)) \end{bmatrix} . \tag{2.71}$$

The constants a_i are given by:

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix} = \begin{bmatrix} \\ (L\varphi_i(t),\varphi_j(t)) \\ \\ \end{bmatrix}^{-1} \begin{bmatrix} (f(t),\varphi_1(t)) \\ (f(t),\varphi_2(t)) \\ \cdot \\ \cdot \\ (f(t),\varphi_n(t)) \end{bmatrix} . \qquad (2.72)$$

where the ij^{th} element of the matrix is $(L\varphi_i(t),\varphi_j(t))$. The matrix will be (k x n) x (k x n) where "k" is the order of the original differential equation and "n" is the order of the approximation.

The approximation given by Equations (2.72) and (2.67) converges to the exact solution as $n \to \infty$. (See Appendix D.) The approximation due to Galerkin converges at the same rate as the mean square "best" approximation to x(t), Atkinson (1).

To apply this technique to the direction cosine equations make the following substitutions

$$A(t) = \frac{d\theta}{dt} \qquad (2.73)$$

$$B(t) = 0$$

Then the development of the technique applies directly.

CHAPTER III

ALGORITHM COMPARISONS

1. Analytical Error Determination

There are two factors which determine the usefulness of an algorithm. The first is accuracy; the second is speed. For accuracy comparisons we will consider three approaches: (1) the Galerkin algorithm, (2) one of the commutative algorithms, and (3) a first-order approximation to the commutative solution. Consider first the first-order approximation of the commutative algorithm. The commutative approximation was given by:

$$C(NT + T) = \exp [\Omega(NT)T] \cdot C(NT) \qquad (3.1)$$

where T is the time step. Expanding the exponential in a Taylor series and truncating to the linear term gives

$$C(NT) = (I + \Omega(NT) \cdot T)\ C(NT) \quad . \qquad (3.2)$$

This is nothing more than a Euler integration of the original differential equation. Now we will consider a special case for the "$\Omega(t)$" matrix which will allow an analytical determination of the error.

For the special case where $\omega_1 = \omega_2 = 0$ and ω_3 is constant, difference equations may be written for the Euler algorithm and the Galerkin algorithm. These may be z-transformed and from the z-transform equations it is possible to determine the output of the algorithm analytically. This method has the advantage of showing explicitly what effects the driving frequency and time step have on the integration accuracy. It should be noted that a full evaluation of the exponential would give zero error for this case since $\omega(t)$ is constant and therefore, the " " matrix is truly commutative.

Let $\omega_1 = \omega_2 = 0$ and ω_3 be constant

$$\omega \triangleq \omega_3 \quad . \tag{3.3}$$

Consider the first column of the cosine matrix

$X(N) \triangleq C_{11}$ at the N^{th} integration step

$Y(N) \triangleq C_{21}$ at the N^{th} integration step

and C_{31} will not be considered since it does not change. The Euler Algorithm is:

$$\begin{aligned} X(N+1) &= X(N) + \omega\, T\, Y(N) \\ Y(N+1) &= Y(N) - \omega\, T\, X(N) \end{aligned} \tag{3.4}$$

where T is the integration step size. The z-transformed equations are (X and Y are functions of z)

$$zX + zX_o = X + \omega\, YT$$

$$zY + zY_o = Y - \omega\, XT \quad .$$

X is then given by

$$X = \frac{-z(z-1)X_o - z\omega T Y_o}{z^2 - 2z + 1 + \omega^2 T^2} \quad . \tag{3.5}$$

The inverse z-transform of the above equation is:

$$X(N) = a^N \ (-X_o \cos Nb - Y_o \sin Nb) \tag{3.6}$$

where

$$\begin{aligned} a &\triangleq \sqrt{1 + \omega^2 T^2} \\ b &\triangleq \tan^{-1} \omega T \quad . \end{aligned} \tag{3.7}$$

The exact solution is known to be of the form

$$X(t) = -X_o \cos \omega_s t - Y_o \sin \omega_s t \quad . \tag{3.8}$$

In order that the numerical routine gives exact answers, then the constants must be given by:

$$a = 1 \qquad\qquad b = \omega_s T \quad . \tag{3.9}$$

The deviation of "a" from unity is seen to cause the numerical solution to always diverge. When the ωT product is small, however, the rate of divergence is small enough to be acceptable. The constant b is best examined by expanding the arctangent function in a Taylor series. The frequency of the algorithm output is then given by:

$$\begin{aligned} \omega_s &= \frac{1}{T}\left(\omega T - \frac{\omega^3 T^3}{3} + \frac{\omega^5 T^5}{5} - \frac{\omega^7 T^7}{7} + \ldots\right) \\ &= \omega - \frac{\omega^3 T^2}{3} + \frac{\omega^5 T^4}{5} - \frac{\omega^7 T^6}{7} + \ldots \end{aligned} \tag{3.10}$$

This indicates that the error in frequency will be intolerable when the ωT product is larger than unity.

Now consider the Galerkin algorithm. The difference equations are given by:

$$X_{(N+1)} = X(N) + \omega T \frac{[Y(N) - 2/3\ \omega T\, X(N)]}{[1 + (2/3)^2\ \omega^2\ T^2]}$$

$$Y_{(N+1)} = Y(N) + \omega T \frac{[-X(N) - 2/3\ \omega T\, Y(N)]}{[1 + (2/3)^2\ \omega^2 T^2]} \quad . \tag{3.11}$$

Let

$$K = \frac{\omega T}{[1 + (2/3)^2 \omega^2 T^2]} \quad .$$

The z-transformed equations are:

$$zX + zX_o = X + KY - 2/3\ \omega TKX$$

$$zY + zY_o = Y - KX - 2/3\ \omega TKY \quad . \tag{3.12}$$

X is then given by

$$X = \frac{-zX_o[z - (1 - 2/3\ \omega TK)] - zKY_o}{[z^2 - 2z(1-2/3\ \omega TK) + K^2 + (1 - 2/3\ \omega TK)^2]} \quad . \tag{3.13}$$

Again, X_N is given exactly by Equation (3.8). Now

$$a = \sqrt{\frac{1 - 1/3\ (\omega T)^2}{(1 + (2/3)^2\ \omega^2 T^2)}} \tag{3.14}$$

and

$$b = \tan^{-1} \frac{K}{1 - 2/3\ \omega TK} \quad .$$

It is apparent that a becomes exact (i.e., unity) for $\omega T = 0$ as before, but now it becomes less than one for a large ωT product. This means that the solution is always bounded.

$$\omega_s = \frac{1}{T}\frac{\omega T}{(1 - 2/3\ \omega T)^3} - \frac{\omega^3 T^3}{3(1 - 2/3\ \omega T)^3} + \frac{\omega^5 T^5}{5(1 - 2/3\omega T)^3} + \ldots$$

$$\omega_s = \frac{\omega}{(1 - 2/3\omega T)^3} - \frac{\omega^3 T^2}{3(1 - 2/3\omega T)^3} + \frac{\omega^5 T^4}{5(1 - 2/3\omega T)^3} + \ldots \qquad (3.15)$$

When ωT is in the neighborhood of one or larger, the frequency error is large. A plot of "a" and "b" versus ωT for each of the algorithms is shown in Figures 7 and 8. These plots show that the solution generated by the Euler technique always has a diverging exponential envelope, whereas, the envelope for the Galerkin technique converges.

2. Error for a Coning Motion

One motion which is known to present a problem for the commutative type algorithm is the coning motion. This occurs when one axis of the body-fixed frame moves around in a cone. Consider Figure 9a. This figure depicts a disk on a shaft which is being turned at some constant angular velocity "ω". The disk is in contact with a plane at its bottom most point. There is no slip between the disk and the plane. The disk then travels around a circle in the plane as it turns around

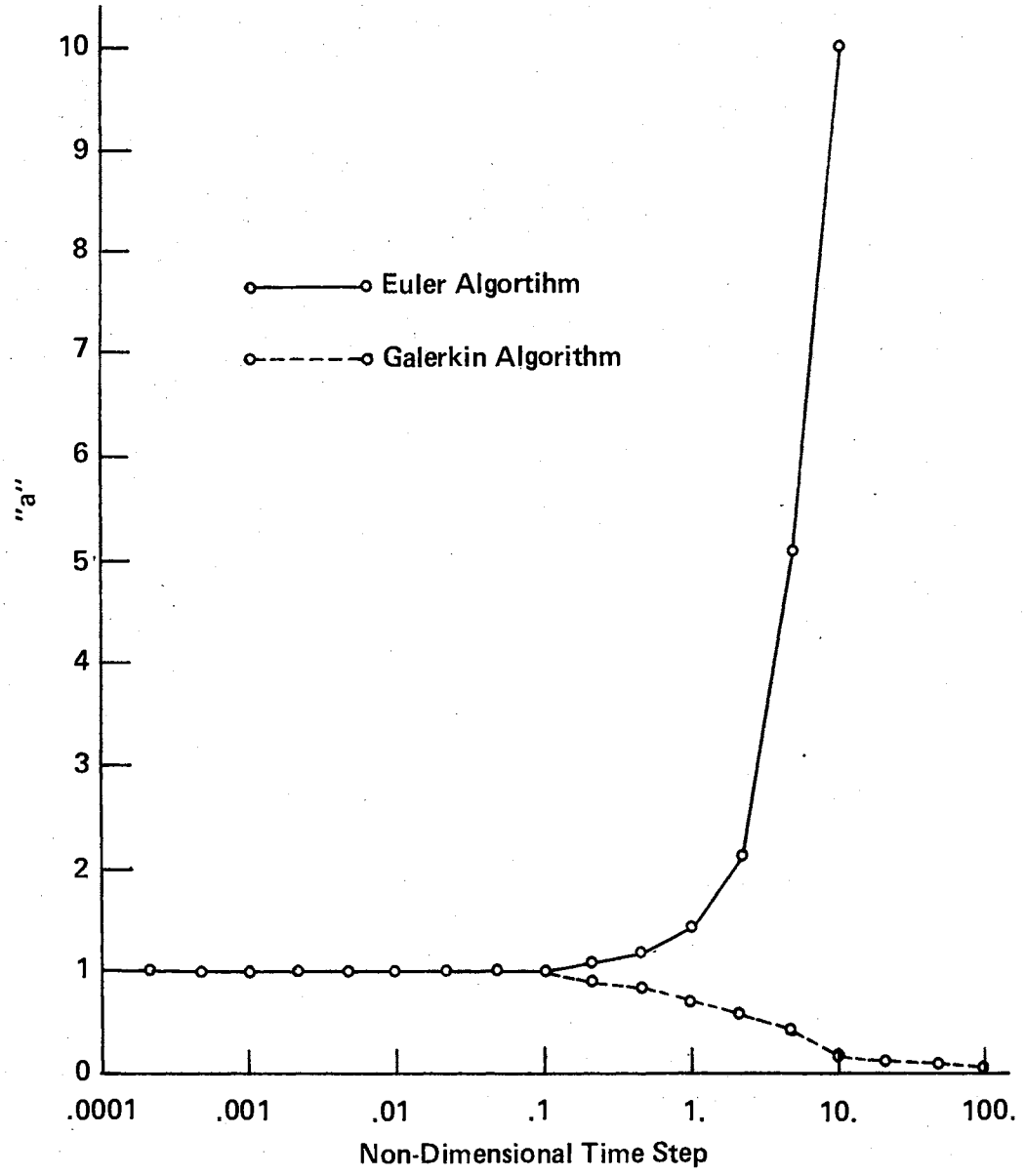


Figure 7. Comparison of Amplitude Envelope for the Euler Algorithm and the Galerkin Algorithm

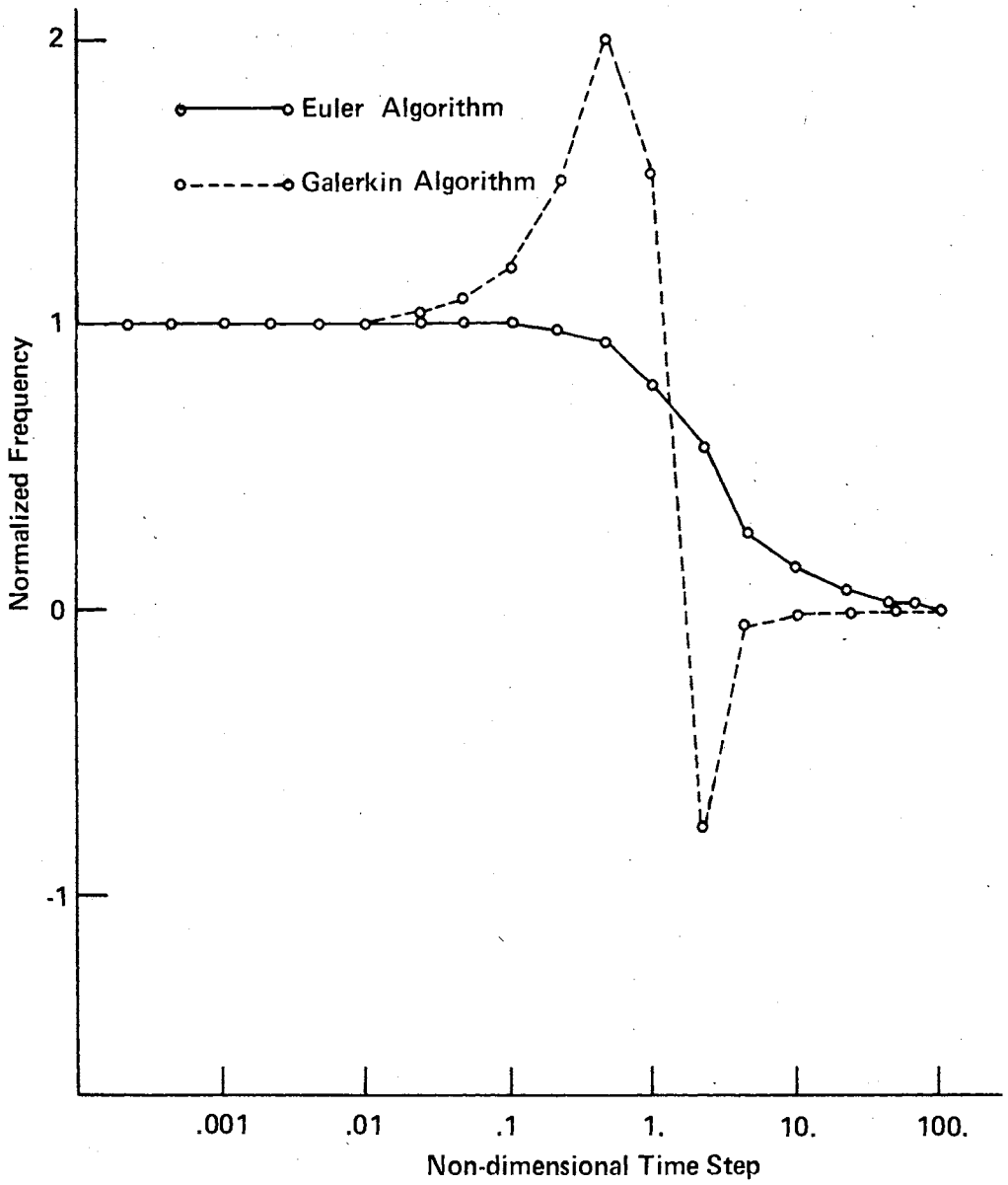
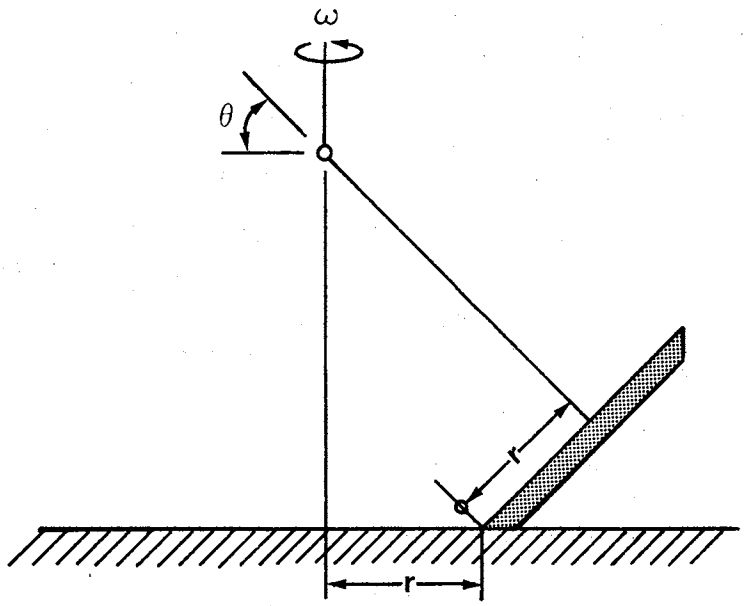
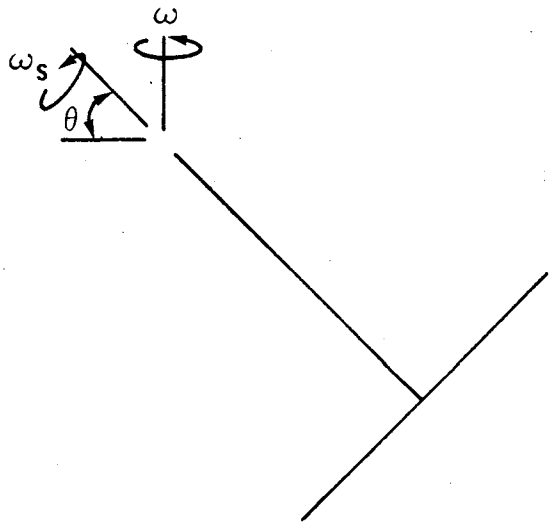


Figure 8. Comparison of Frequency for the Euler Algorithm and the Galerkin Algorithm



a.) No-slip, Equal-rate Coning Motion



b.) General Coning Motion

Figure 9. Illustration of Coning Motion

its central axis. An axis system is fixed to the disk so that the xy plane is the plane of the disk and the y axis initially goes through the point of contact between the disk and the plane. The z axis is coincident with the shaft.

Since the diameter of the rotating disk and the diameter of the circle it describes on the plane are the same, then the orientation of the disk after one full revolution will be the same as at the start. Fix a coordinate frame to the disk so that the z axis is along the driving shaft and the x and y axes are in the plane of the disk. The angular rates about these axes as the disk moves around can be found to be:

$$\begin{aligned} \omega_x &= -\omega \cos\phi \sin\theta \\ \omega_y &= -\omega \cos\phi \cos\theta \\ \omega_z &= \omega (1 + \sin\theta) \end{aligned} \tag{3.16}$$

where $\theta = \omega t$. The initial condition is:

$$C(0) = I \quad . \tag{3.17}$$

More generally, we can consider the case (depicted in Figure 9b) where the axis system rotating about a fixed line (at an angle of $\phi + (\pi/2)$ to its "z" axis) with an angular rate "ω" and about its "z" axis with an angular rate "ω_s". It is clear that if:

$$\omega_s = \omega$$

this case degenerates to the previous one. If ω_s is different from ω, then the disk in Figure 9a slips as it moves

around in a circle. The angular rates for the general coning motion are given by:

$$\begin{aligned} \omega_x &= -\omega \cos\phi \sin \\ \omega_y &= -\omega \cos\phi \cos \\ \omega_z &= \omega_s + \omega \sin\phi \end{aligned} \tag{3.18}$$

where $ = \omega_s t$.

By considering sets of equivalent single axis rotations and the direction cosines related to each rotation, it is possible to find a solution for the general coning motion. The cosine matrix for the coning motion is then given by:

$$C = C(0,0,\psi)\ C(\phi,0,0)\ C(0,-\theta,0)\ C(-\phi,0,0)$$

$$= \begin{bmatrix} \cos\theta\cos\psi - \sin\theta\sin\ \sin\phi & & \\ -\sin\ \cos\theta - \cos\ \sin\theta\sin\phi & & \\ -\sin\theta\cos\phi & & \end{bmatrix}$$

$$\begin{bmatrix} \cos\psi\ \sin\theta\sin\phi + \sin\ \cos^2\phi + \sin\ \cos\theta\sin^2\phi \\ -\sin\psi\ \sin\theta\sin\phi + \cos\psi\ \cos^2\phi + \cos\ \cos\theta\sin^2\phi \\ -\sin\phi\cos\phi + \cos\phi\sin\phi\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\psi\ \sin\theta\cos\phi - \sin\ \sin\phi\cos\phi + \sin\ \cos\theta\sin\phi\cos\phi \\ -\sin\psi\ \sin\theta\cos\phi - \cos\psi\ \sin\phi\cos\phi + \cos\psi\ \cos\theta\sin\phi\cos\phi \\ \sin^2\phi + \cos\theta\cos^2\phi \end{bmatrix}. \tag{3.19}$$

Now Equation (3.19) is actually the solution to the time-varying homogeneous system

$$\dot{C}(t) = \Omega(t)\ C(t)$$

where "$\Omega(t)$" is the skew symmetric matrix of angular rates. The angular rates composing $\Omega(t)$ are those for the general coning motion. The solution derived above is for

$$C(0) = I$$

and thus, is the state transition matrix $\Phi(t_2,t_1)$ for the system. The solution at any time may then be expressed as:

$$C(t_2) = \Phi(t_2,t_1)\, C(t_1) \quad . \tag{3.20}$$

It is easily demonstrated that in order to be a state transition matrix, it must satisfy

$$\frac{d}{dt_2}\, \Phi(t_2,t_1) = \Omega(t_2)\, \Phi(t_2,t_1) \tag{3.21}$$

and

$$\Phi(t_1,t_1) = I \quad . \tag{3.22}$$

The solution presented above satisfies Equations (3.21) and (3.22). It is of academic interest to note that the solution obeys Floquet's theorem for periodically varying systems. In order to test the convergence of the Galerkin algorithm, we will consider two special cases of the coning motion. The first is the single-axis rotation. This is a constant rate rotation about one axis of the reference frame. The conditions for this type of motion are described by

$$\phi = \pi/2 \quad , \quad \omega = 0 \quad , \quad \omega_s = \text{constant} \quad . \tag{3.23}$$

The other type of motion is a 90° cone for which the conditions are

$$\phi = 0 \quad , \quad \omega = \text{constant} \quad , \quad \omega_s = \omega \quad . \tag{3.24}$$

These cases have been solved using a Galerkin approximation. Three choices of basis functions were used. First sine-cosine basis functions were chosen. By using an appropriate period for the circular functions, the solutions to both the single axis rotation and the ninety degree cone have an exact representation as a linear combination of a finite number of basis functions. The set of sine-cosine functions used is:

$$\{1, \sin \omega t, \cos \omega t, \sin 2\omega t, \cos 2\omega t, \ldots\} \quad . \tag{3.25}$$

The single axis rotation converges exactly at three terms and the ninety degree cone at five.

The use of trigonometric basis functions as a representation is unrealistic, however, since, in general, we will not know the period of motion in advance. In a more realistic approach, polynomials are used to do the approximation. Two sets of polynomial basis functions are used, Legendre polynomials because of their orthogonality, and a simple power series because of its simplicity. Using Legendre polynomials gives the basis set:

$$\{P_0, P_1, P_2, \ldots\}$$

where the P_i are defined by the recursion

$$P_0(x) = 1$$

$$P_1(x) = x$$

and

$$P_{k+1}(x) = \frac{(2k + 1) \cdot P_k(x) - kP_{k-1}(x)}{k + 1}$$

The basic set for the simple power series is:

$$\{1, t, t^2, \ldots\} \quad , \tag{3.26}$$

The advantage of using orthogonal polynomials is that the conditioning on the matrix $(L\phi_j, \phi_i)$ is better than when the simple power series is used.

In order to measure conditioning on a matrix, a condition constant may be defined by:

$$\text{Cond } (A) = \|A\| \cdot \|A^{-1}\| \tag{3.27}$$

where the matrix norm of "A" is defined by:

$$\|A\| = \max_i \sum_j |a_{ij}| \quad . \tag{3.28}$$

If A = I, then Cond (A) = 1 and as A becomes increasingly ill-conditioned, then Cond (A) becomes large. Experience has shown that condition constants of magnitude 10^{10} will cause a standard matrix inversion routine (Gauss-Siedell) to begin to experience under flows during some of its calculations on an IBM 360 using double precision arithmetic.

Figure 10 shows a comparison of condition constants for the $(L\phi_j, \phi_i)$ matrix using Legendre polynomials and the simple power series basis functions. At high orders it can be seen that the condition constant for the simple power series becomes quite large. For the range considered, however, the

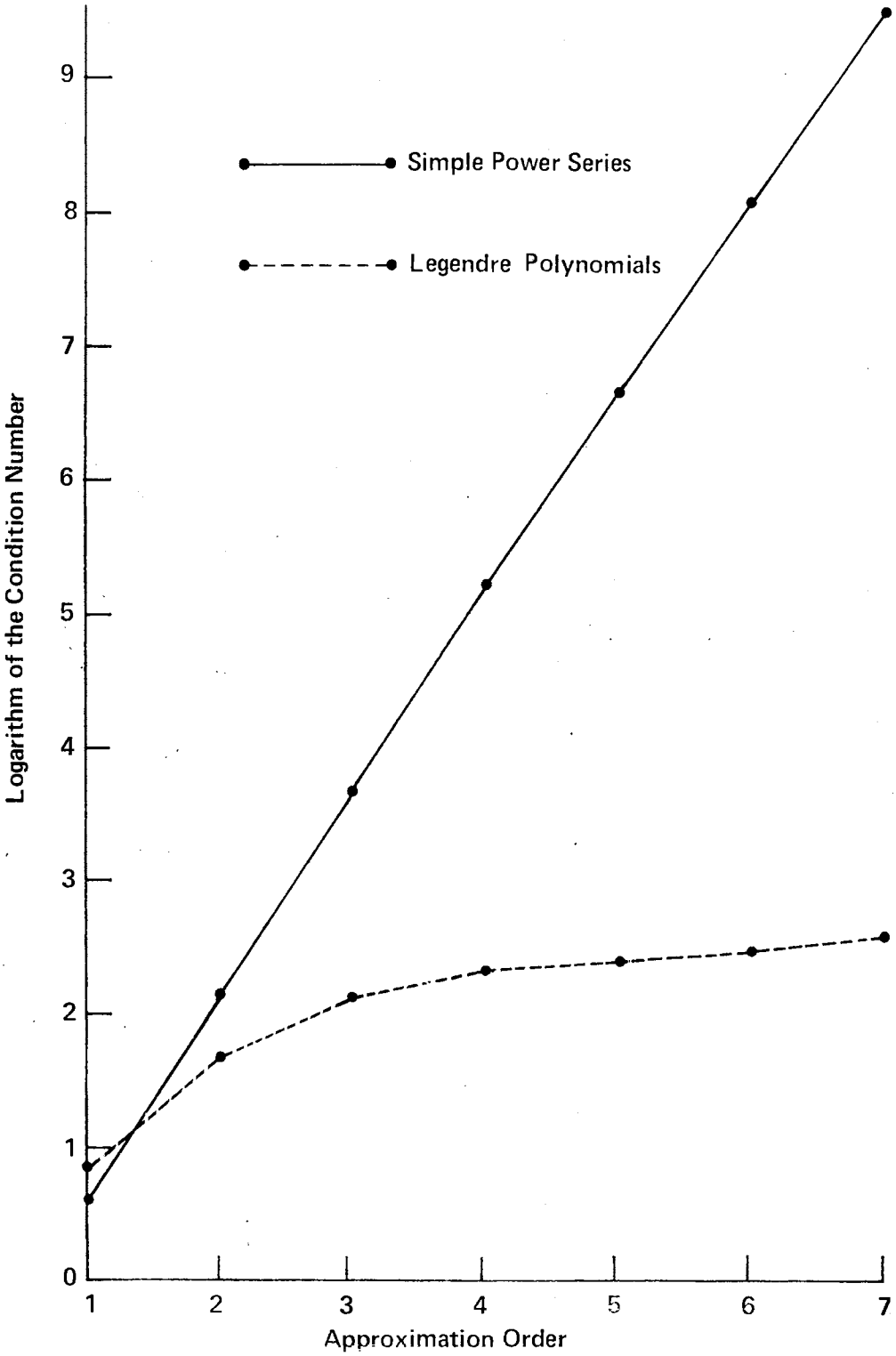


Figure 10. Comparison of Condition Numbers for Legendre Polynomials and Simple Power Series

difference in the answers generated by the two sets were, for all practical purposes, the same. Apparently the answer is fairly independent of the conditioning on the matrix.

In order to measure the error in making the approximation, it is necessary to specify a suitable norm for the error function. In general, a norm for the space of "p" integrable functions on [a,b] (i.e., $L^p(a,b)$) is given by

$$\|\cdot\|_p = \left[\int_a^b |\cdot|^p \, dt \right]^{1/p} . \tag{3.29}$$

We shall consider two cases:

$$p = 2$$

$$p = \infty \quad .$$

The case p = 2 corresponds to the mean square norm. The case where $p = \infty$ corresponds to the uniform norm. The uniform norm may be alternatively defined by:

$$\|\cdot\|_\infty = \max_{a \le t \le b} \|\cdot\| \quad . \tag{3.30}$$

If we specify that the approximation converges uniformly as $N \to \infty$, this implies that:

$$\lim_{N \to \infty} \|e\|_\infty = 0 \tag{3.31}$$

where e(t) is the approximation error for $t \in [a,b]$. Uniform convergence is a stronger condition than mean square convergence. For instance, a sequence of continuous functions can converge to a discontinuous function in the mean square

sense, but will not converge uniformly. If the function being approximated is smooth, then mean square convergence implies uniform convergence, Kantorovich (22).

Figures 11, 12, 13 and 14 show the norm of the approximation error for the direction cosines using trigonometric basis functions and using polynomial basis functions. The approximation error is actually a k x k matrix valued function. The norm of each function in the matrix is taken in two ways, the mean square norm and the uniform norm. The norm of the resulting matrix is then defined by the taxicab norm as above. The norms of the matrix error function are then:

$$\|[e_{ij}(t)]\|_{2}^{k \times k} = \max_{i} \sum_{j} \|e_{ij}(t)\|_{2} \qquad (3.32)$$

and

$$\|[e_{ij}(t)]\|_{\infty}^{k \times k} = \max_{i} \sum_{j} \|e_{ij}(t)\|_{\infty} \quad . \qquad (3.33)$$

Accuracy of Galerkin's technique may also be improved by taking smaller steps. To demonstrate this, several runs were made using first, second and third-order approximations. The beginning time and final time were held constant for each of the runs. As this technique was applied more often (and the step size reduced accordingly), the results of the

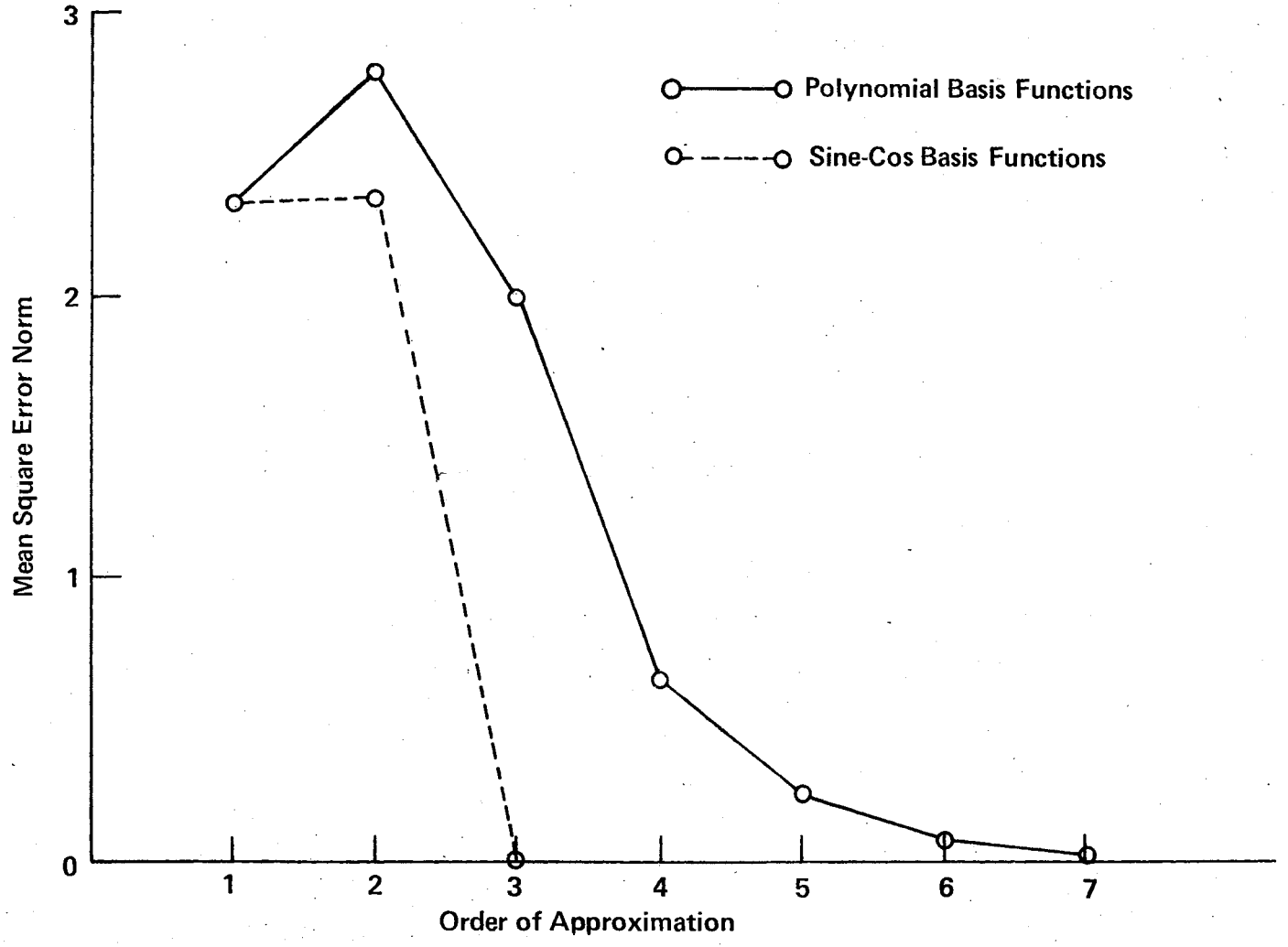


Figure 11. Uniform Norm of the Error for a Single Axis Rotation Using Galerkin's Technique

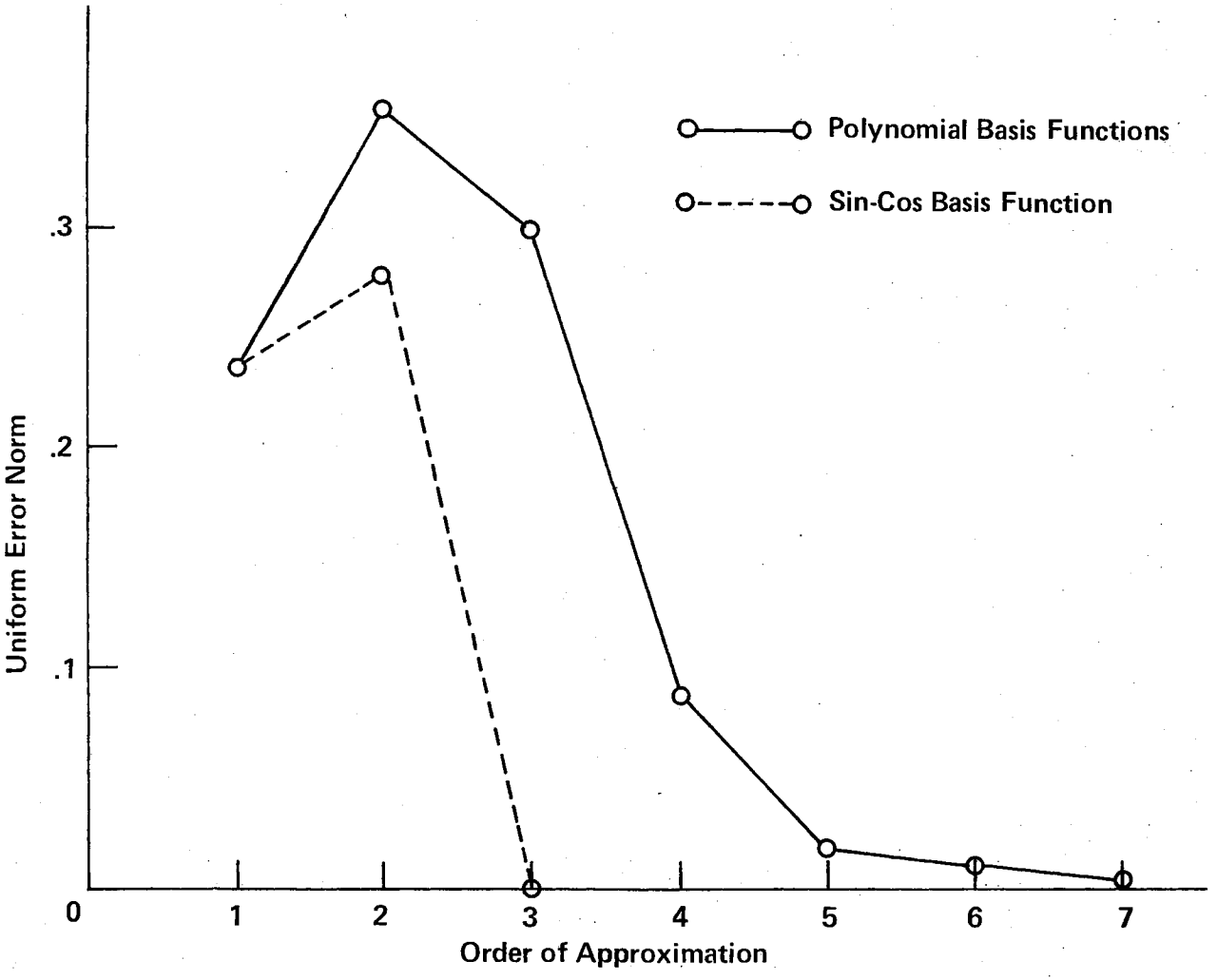


Figure 12. Mean Square Error for a Single Axis Rotation Using Galerkin's Technique

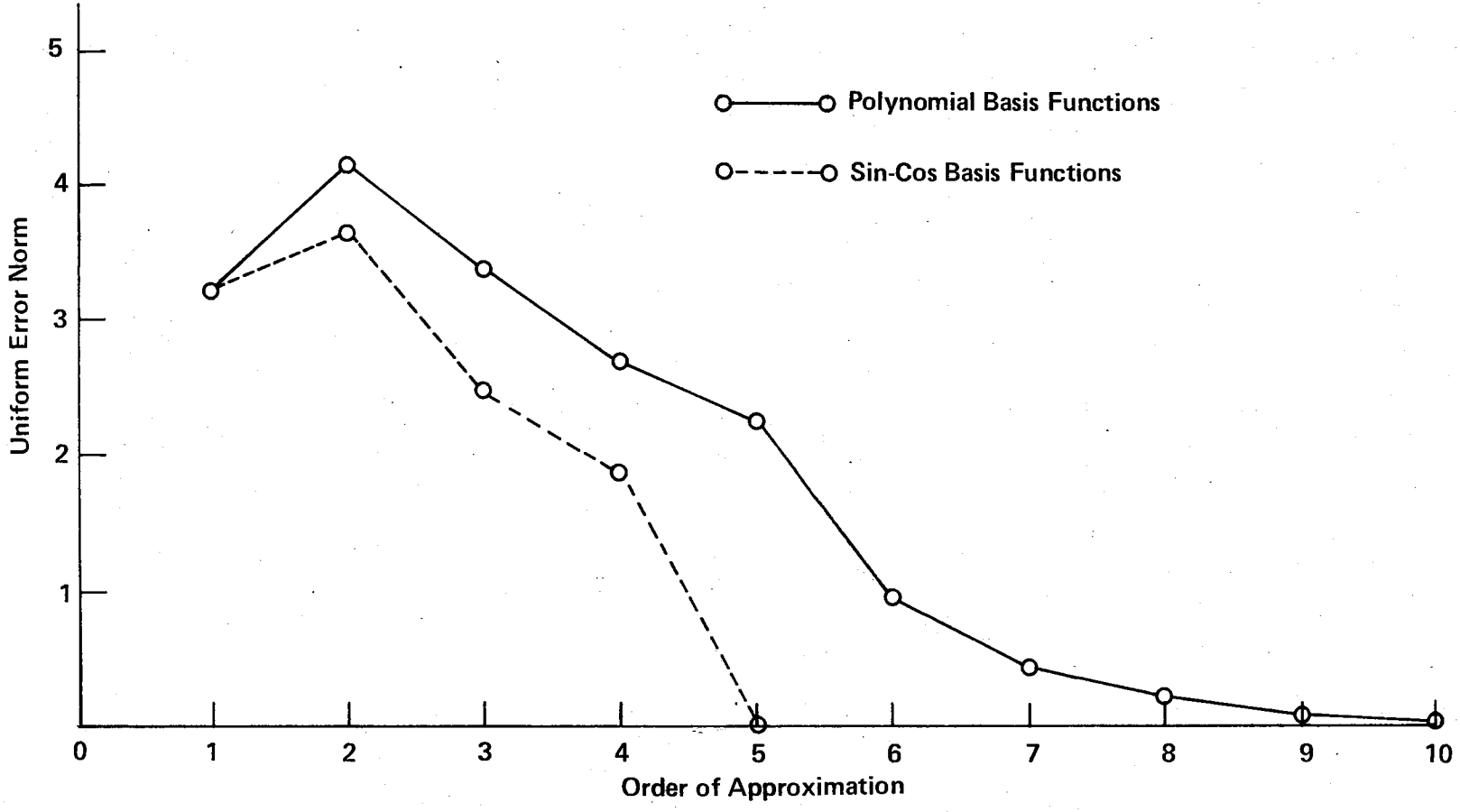


Figure 13. Uniform Norm of the Error for a 90^{o} Cone Using Galerkin's Technique

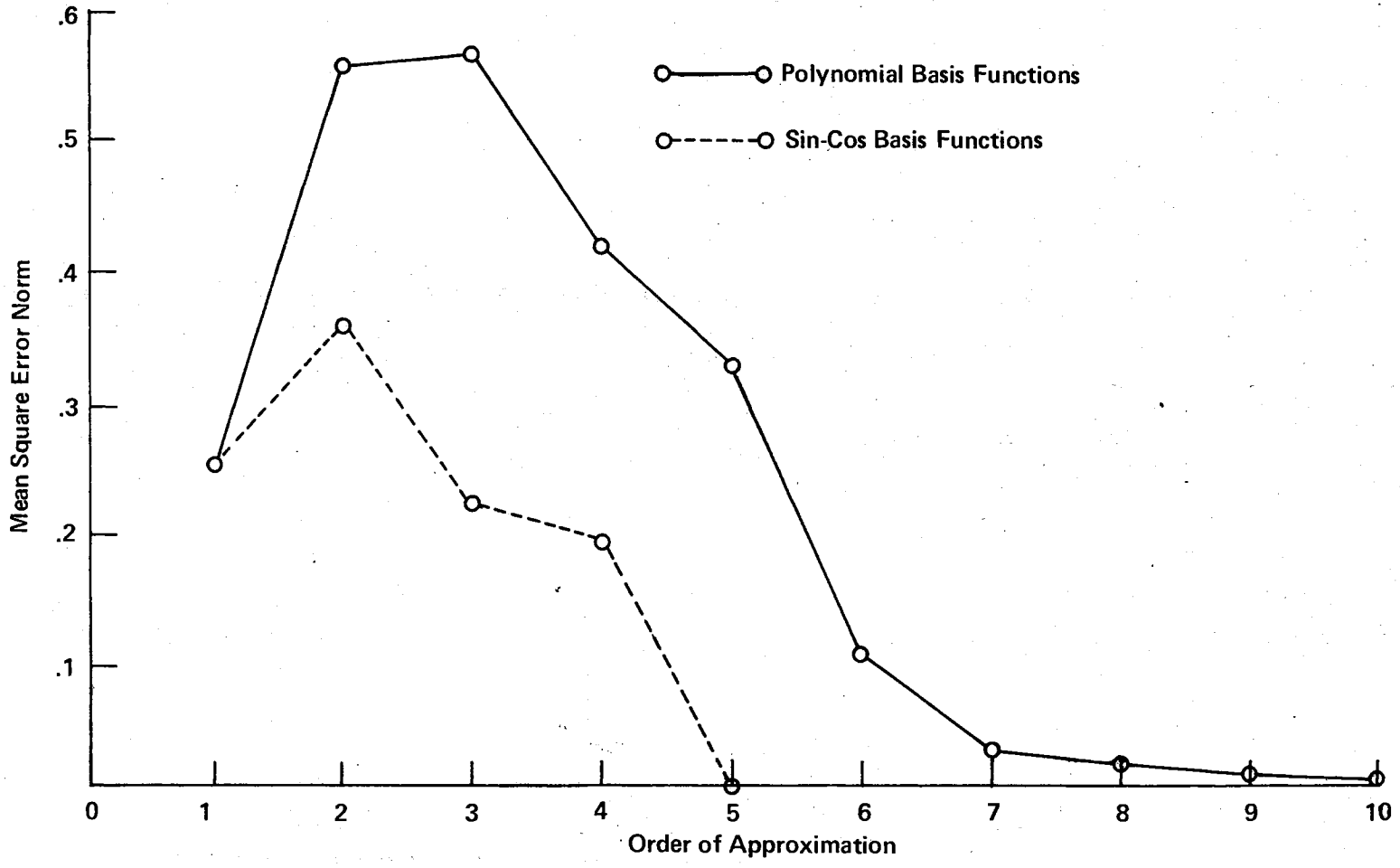


Figure 14. Mean Square Error Norm for a 90^{o} Cone Using Galerkin's Technique

approximation became better as is shown in Figures 15 , 16, 17, 18, 19 and 20.

It has been demonstrated that given an arbitrary angular rate history, we may approximate the direction cosines with arbitrary accuracy by either increasing the order of the approximation or increasing the frequency at which the technique is applied.

In a similar manner, a commutative algorithm may be applied to the problem of propagating the solution to the ninety degree cone. The error incurred by using a commutative algorithm may be decreased by increasing the frequency at which the technique is applied. Figures 18 and 19 show the approximation error using a commutative algorithm to propagate the direction cosines to a ninety degree cone.

These two algorithms are now ready for an important comparison, that is, accuracy versus computer time requirements. Table I gives the times required to apply each algorithm once, that is, perform one update. This table and Figures 11 through 19 show that for the 90^{o} cone, given perfect measurements of the angular rates, the commutative algorithm is more efficient. The question may arise as to whether the 90^{o} cone is a reasonable test case. There are two reasons it was chosen. First, some angular rate history which is not commutative must be chosen in order to test the commutative

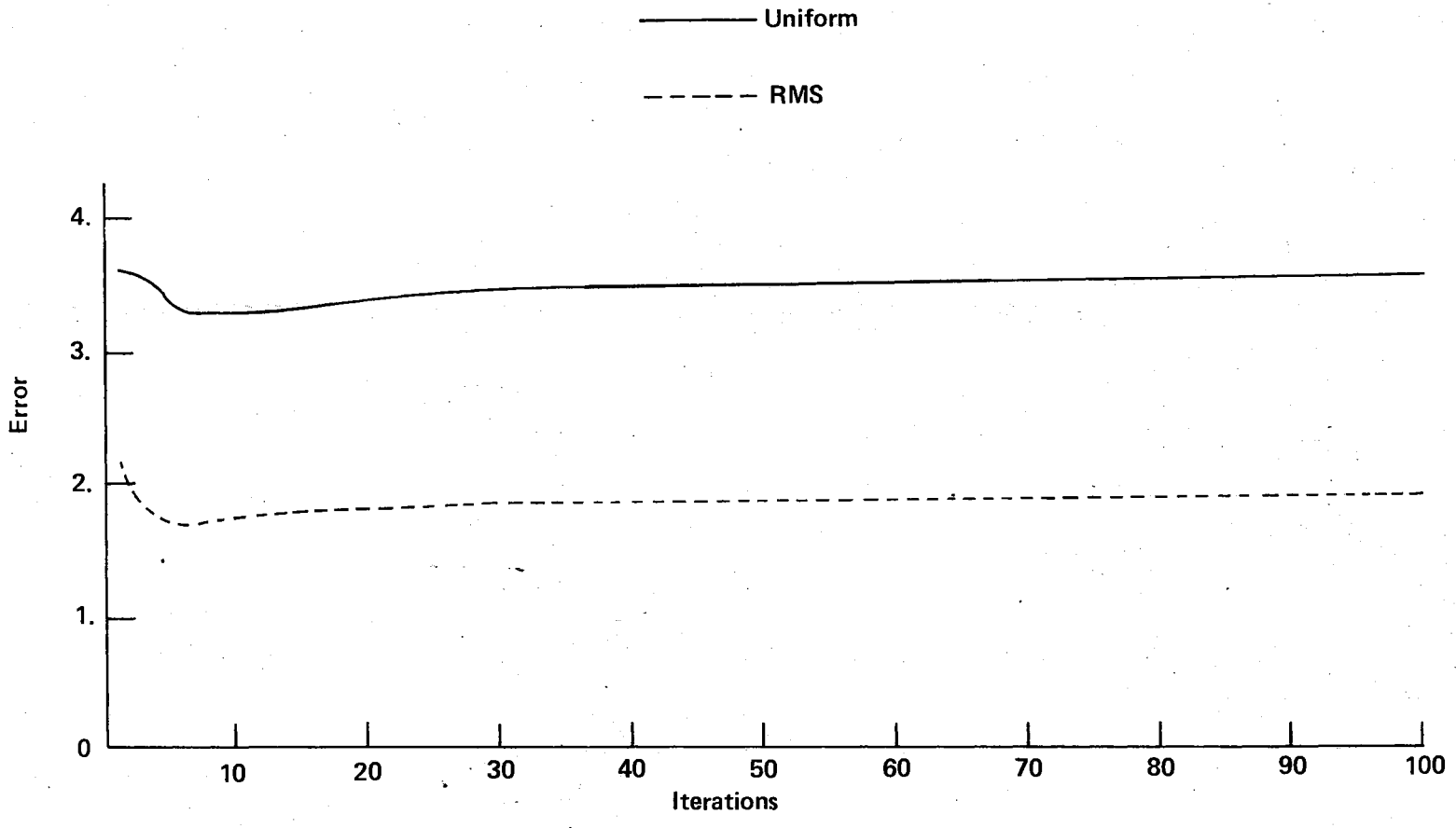


Figure 15. Error for the 90^{o} Cone by Repetitive Application of First-Order Galerkin

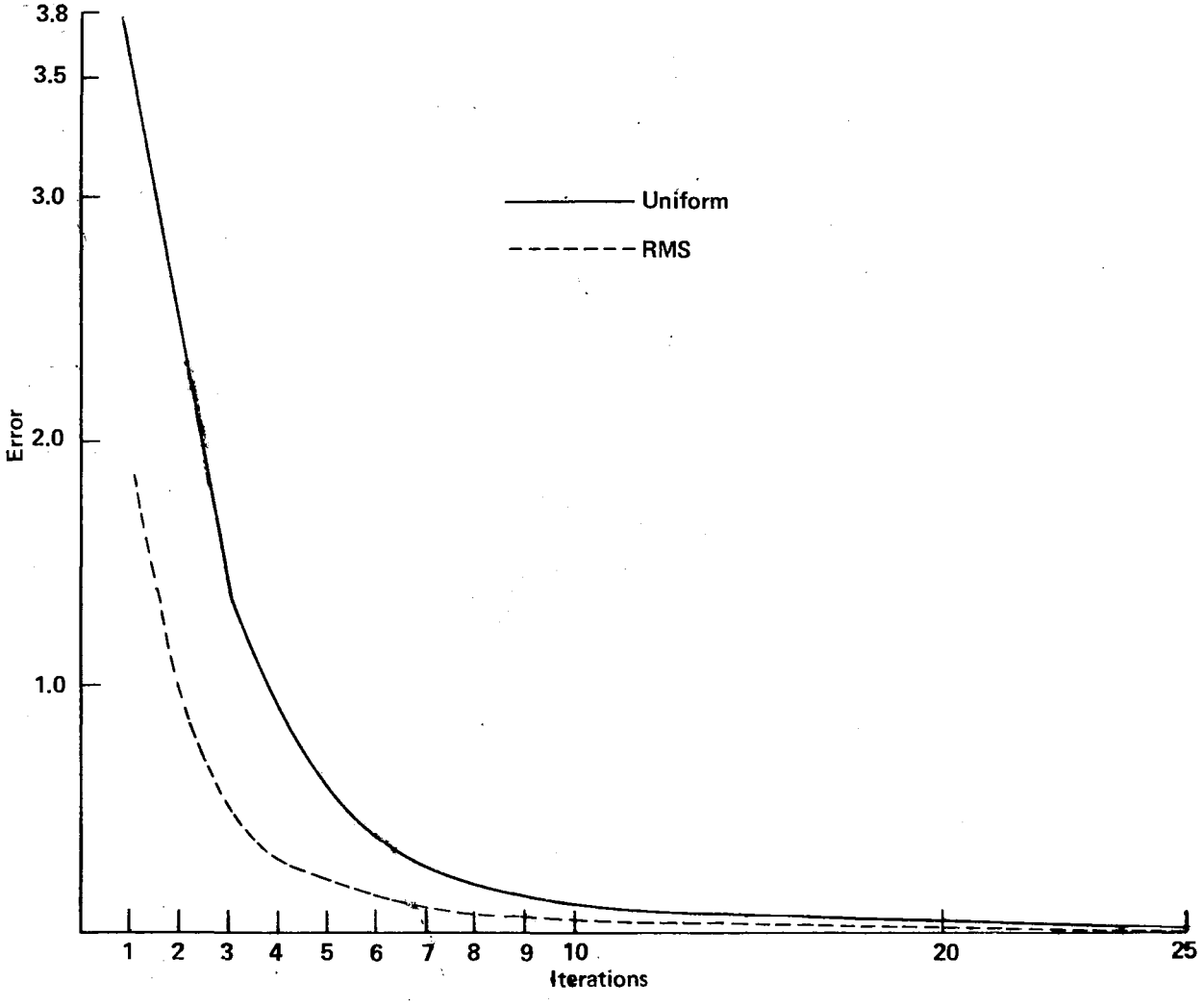


Figure 16. Error for 90° Cone by Repetitive Application of Second-Order Galerkin

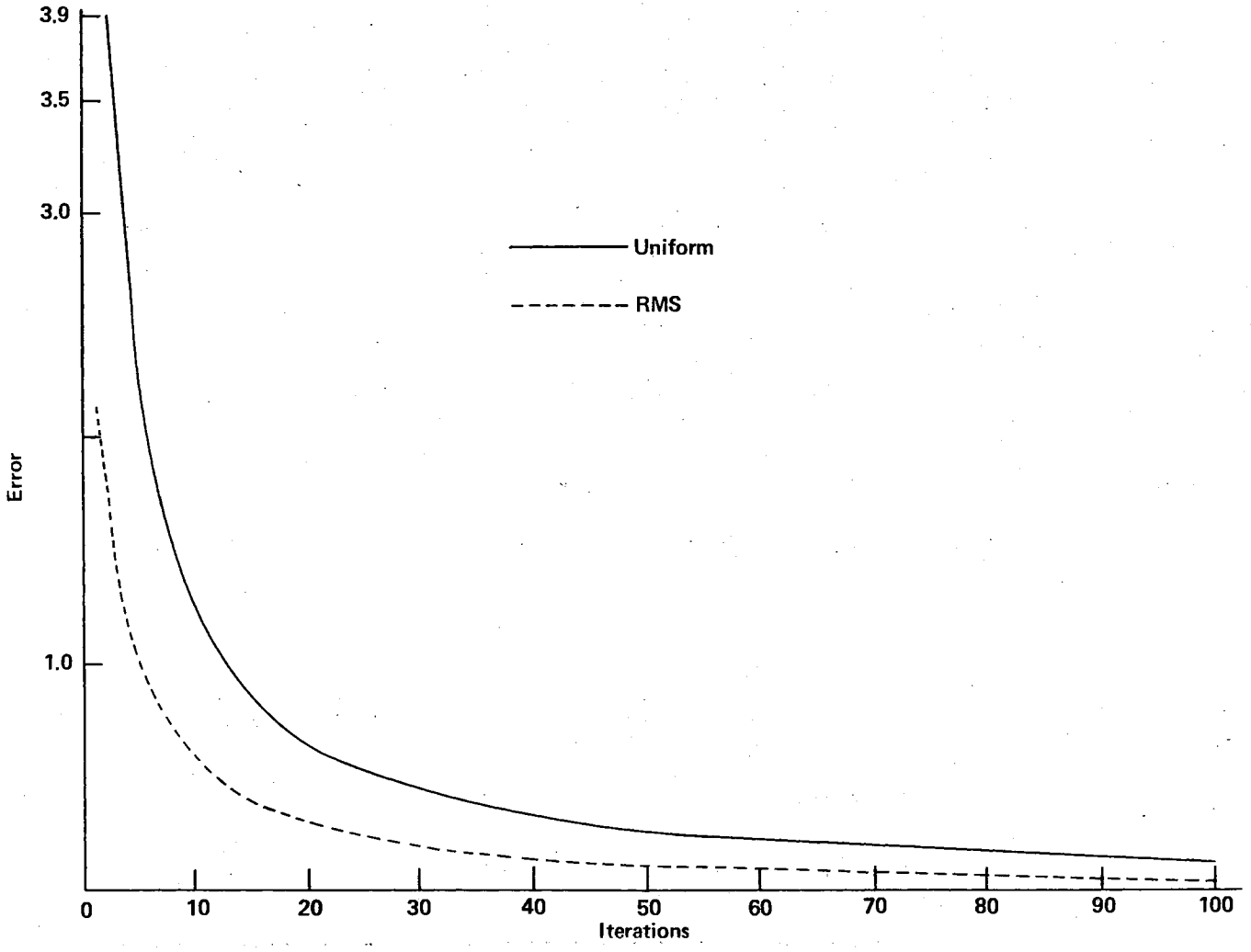


Figure 17. Error for 90° Cone by Repetitive Application of Third-Order Galerkin

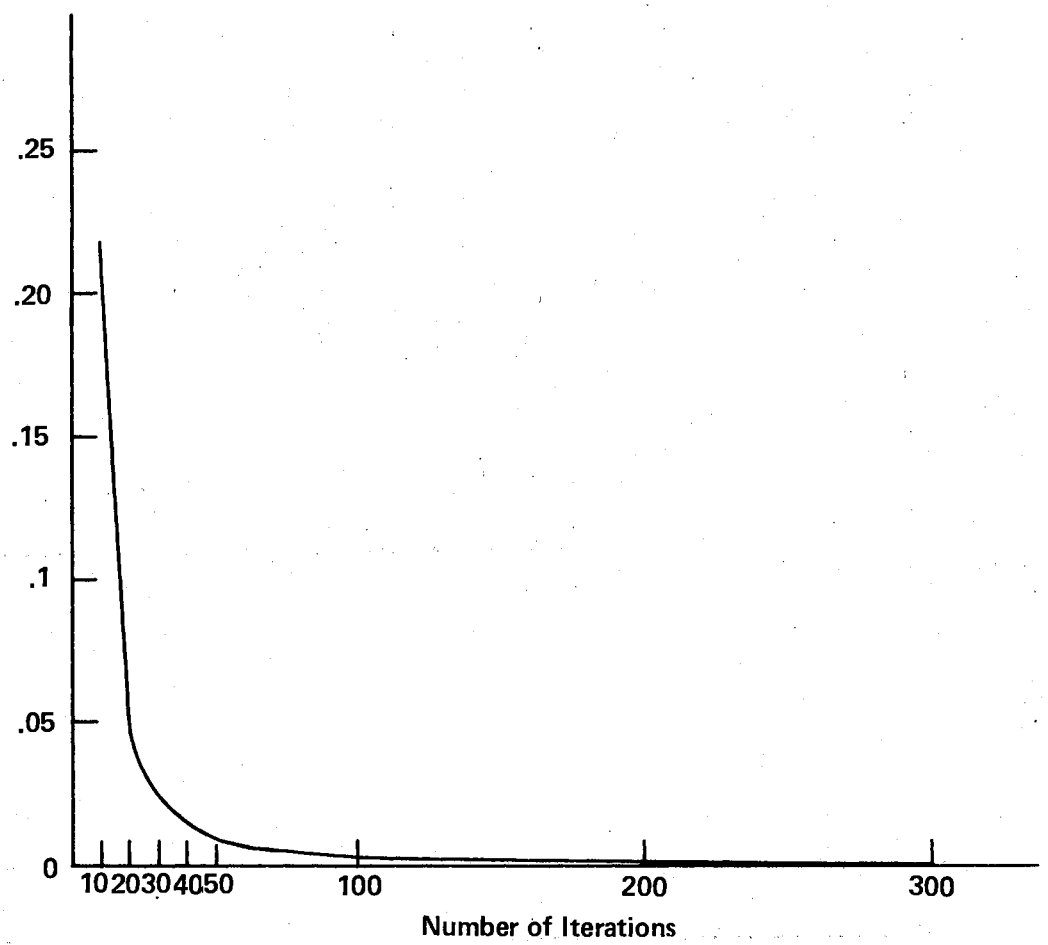


Figure 18. RMS Error Using a Commutative Algorithm for a 90^{o} Cone

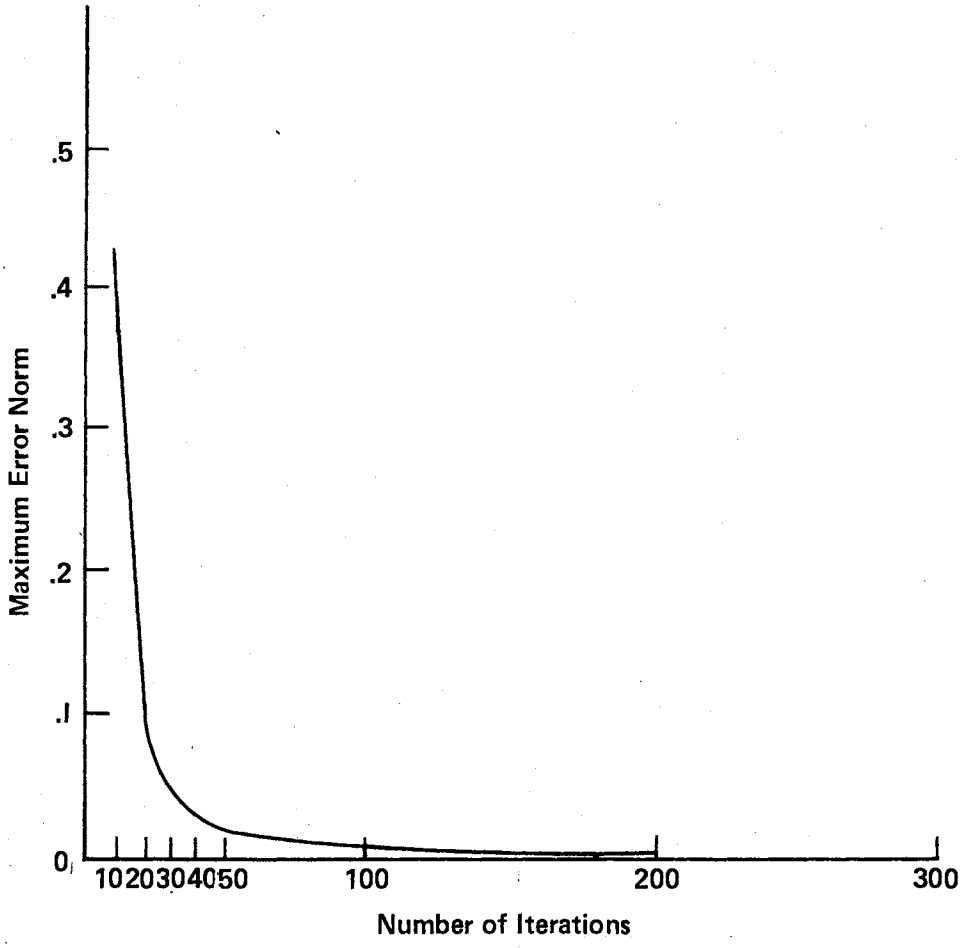
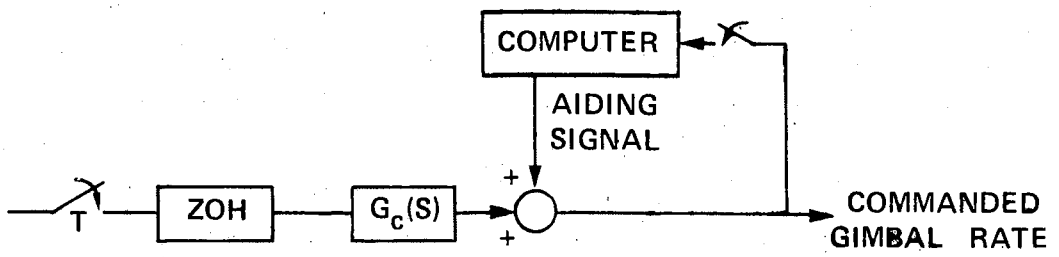
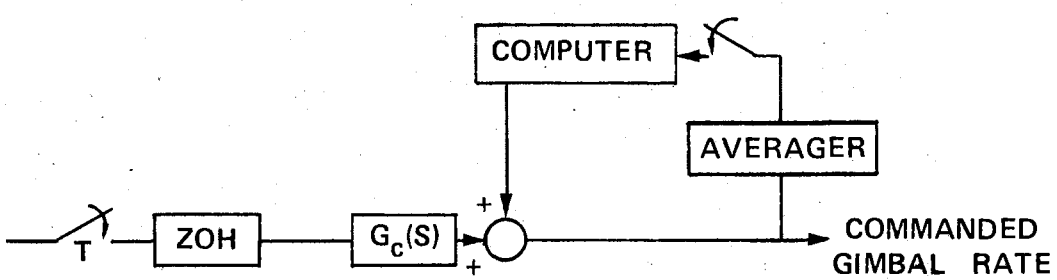


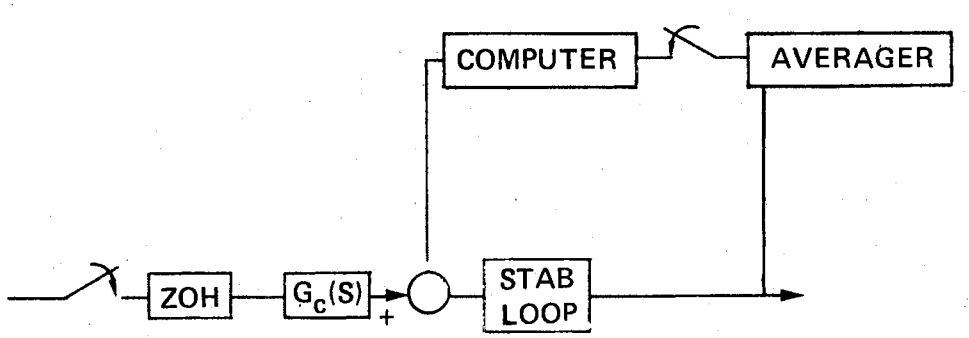
Figure 19. Maximum Error Using a Commutative Algorithm for a 90° Cone



a.) System Now Implemented



b.) Modification To Use New Integration Algorithm



c.) Modification for True Gimbal Rates

Figure 20. APT System Modification to Implement New Cosine Algorithm

TABLE I

COMPUTER TIME REQUIREMENTS

Integration Method	Number of Operations					Total Time
	MULT (9.7 μ sec)	ADD (3. μ sec)	DVD (17 μ sec)	VECTOR ROTATE (40 μ sec)	SQRT (20 μ sec)	
Euler	27	27	0	0	0	342.14
STM	42	30	2	1	1	600.4
Quaternion (Propagate)	19	12	2	1	1	314.3
Convert to DCM	13	15	0	0	0	171.1
Total by Quaternion	32	27	2	1	1	485.4
1st Order Galerkin	38	26	0	0	0	446.6
2nd Order Galerkin	301	250	0	0	0	3,669.7
3rd Order Galerkin	1005	888	0	0	0	12,412.5

algorithms at all. Second, the solution form for the coning motion is known so that a sound error measure is available. Whether or not the cone is representative of real maneuvers is another question. Excepting missile and rocket applications, the coning motion is probably more severe than the types of maneuvers which will be encountered in practice, and is a worst case test. In missile and rocket applications, the actual angular rate histories may be much more complex than that of the 90^{o} coning case. It would not be at all unreasonable for one of these to undergo a coning motion of a more general nature. It would be expected that both of the algorithms presented would have to be applied more frequently. As shown in Appendix D, convergence of either algorithm is assured, but these algorithms are limited by computer speed to some finite application frequency. Regardless of these considerations, however, the 90^{o} coning motion serves as a valid basis for comparison and the results here may be expected to hold true in more complex cases also.

Another consideration is deciding between the two algorithms is sensitivity to noise. This is addressed in the next section.

3. Perturbation Error

3.1 Noise Sensitivity for Galerkin Algorithm

The error analysis for the Galerkin algorithm has so far assumed that the angular rate matrix $\Omega(t)$ is known exactly. In practice, this will not be the case. It is hoped that the Galerkin technique will be insensitive to noise which is of much higher frequency than the basis functions used in the approximation. That this is true is demonstrated in this section. The perturbation in the equation is some noise which is added to the actual angular rates as they are measured. The total error "e_T" in using the Galerkin technique is related to the approximation error e_a which is the error present with good measurements and the perturbation error e_p, which is the amount of change in the approximate answer due to noisy measurements, by:

$$\|e_T\| \leq \|e_p\| + \|e_a\| \quad . \tag{3.34}$$

It was shown in Chapter II that in order to solve for the coefficients in the Galerkin approximation, the linear equation that follows has to be solved:

$$\begin{bmatrix} \\ \\ \\ \ldots(L\phi_j(t),\ \phi_i(t))\ldots \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \\ a \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ (x_o,\ \phi_i(t)) \\ \\ \\ \\ \end{bmatrix} \tag{3.35}$$

where $(L\phi_j(t), \phi_i(t))$ is the ij^{th} element of the matrix and "a" and $(x_o, \phi_i(t))$ are both column vectors. The "L" operator was defined as:

$$L\,x(t) \triangleq I\,x(t) - \int_0^t \Omega(s)\,x(s)\,ds \quad . \tag{3.36}$$

The matrix "$\Omega(s)$" is the skew symmetric matrix of angular rates. In general, the matrix "$\Omega(s)$" is not known in advance, but must be measured as the mission progresses. The measurements are corrupted with noise and this causes some difficulty. Assume that the noise is additive and denote the measured value of the angular rate matrix as "$\bar{\Omega}(s)$", then

$$\bar{\Omega}(s) = \Omega(s) + e(s) \tag{3.37}$$

where e(s) is a skew symmetric matrix of noise terms. The "L" operator is changed and denoting the noisy operator as "$\bar{L}$" gives

$$\begin{aligned} \bar{L}\,x(t) &= I\,x(t) - \int_0^t \bar{\Omega}(s)\,x(s)\,ds \\ &= I\,x(t) - \int_0^t [\Omega(s) + e(s)]\,x(s)\,ds \\ &= L\,x(t) - \int_0^t e(s)\,x(s)\,ds \quad . \end{aligned} \tag{3.38}$$

Now the use of the noisy measurements in the approximation equation gives

$$\left[(L\phi_j(t),\ \phi_i(t)) - \left(\int_0^t e(s)\ \phi_j(t)\ ds,\ \phi_i(t)\right)\right]\left[\tilde{a}\right] = \left[(x_o,\ \phi_i(t))\right] \tag{3.39}$$

where $[\tilde{a}]$ is the vector coefficients determined with noisy measurements. Let

$$L_{ij} = \left[(L\ \phi_j(t),\ \phi_i(t))\right]$$
$$P = \left[\int_0^t e(s)\ \phi_j(s)\ ds,\ \phi_i(t)\right] \tag{3.40}$$

and the vectors $\tilde{A}$ and f be given by

$$\tilde{A} = [\tilde{a}]$$

and

$$f = [(x_o,\ \phi_i(t))] \quad . \tag{3.41}$$

Reasonable statistics for e(t) are:

$$E\{e(t)\} = 0$$
$$E\{e(t)\ e(t+\tau)\} = R_{ee}(\tau) \quad . \tag{3.42}$$

Now direct attention to finding $E\{P\}$ and $E\{P^2\}$.

$$E\{P\} = E\left\{\int_0^1 \int_0^t e(s)\ \phi_j(s)\ ds\ \phi_i(t)\ dt\right\} \quad .$$
$$E\{P\} = \int_0^1 \int_0^t E\{e(s)\}\ \phi_j(s)\ \phi_i(t)\ ds\ dt \quad . \tag{3.43}$$

Since $E\{e(s)\} = 0$ (zero mean) then:

$$E\{P\} = 0 \quad .$$

The variance of P is then defined by:

$$E\{P^2\} = E\left\{ \int_0^1 \int_0^r e(s)\,\phi_j(s)\,ds\,\phi_i(r)\,dr \cdot \int_0^1 \int_0^\rho e(\tau)\,\phi_j(\tau)\,d\tau\,\phi(\rho)\,d\rho \right\} . \qquad (3.44)$$

A change in the order of integration gives

$$E\{P^2\} = E\left\{ \int_0^1 \int_0^1 \int_0^r \int_0^\rho e(s)e(\tau)\phi_j(s)\phi_j(\tau)\phi_i(\rho)ds d\tau dr d\rho \right\}$$

$$= \int_0^1 \int_0^1 \int_0^r \int_0^\rho E\, e(s)e(\tau)\,\phi_j(s)\phi_j(\tau)\phi_i(r)\phi_i(\rho)ds d\tau dr d\rho \qquad (3.45)$$

since the expected value operator and the integral commute. The autocorrelation $R_{ee}(\tau)$ is defined by Equation (3.42). Substitution into the expression above gives

$$E\{P^2\} = \int_0^1 \int_0^1 \int_0^r \int_0^\rho R_{ee}(s-\tau)\phi_j(s)\phi_j(\tau)\phi_i(r)\phi_i(\rho)ds d\tau dr d\rho . \qquad (3.46)$$

In the general case, this expression is quite difficult to integrate as it stands. Some insight into the effect of frequency separation between the basic functions and the noise source can be derived from it, however. To accomplish this, consider the use of sine-cosine basis functions which are orthogonal on [0,1]. With little loss of generality the autocorrelation may be expressed as a Fourier series on the interval $\tau \in [0,1]$

$$R_{ee}(\tau) = a_o + \sum_{n=1}^{\infty} a_n \cos(2\pi n w \tau) + \sum_{m=1}^{\infty} b_m \sin(2\pi n w \tau). \quad (3.47)$$

The autocorrelation function must be restricted to the class of continuous functions for this expression to be exact. The autocorrelation $R_{ee}(\tau)$ must be defined on $\tau \in [-1,1]$ for use in the integral expression for the error variance. Equation (3.47) is only valid on $\tau \in [0,1]$. In order to use it for $\tau \in [-1,1]$, recognize that a property of the autocorrelation function is that

$$R_{ee}(\tau) = R_{ee}(-\tau) \quad . \quad (3.48)$$

This gives the expansion for $R_{ee}(\tau)$ to be:

$$R_{ee}(\tau) = a_o + \sum_{n=1}^{\infty} a_n \cos(2\pi n \tau) + \sum_{m=1}^{\infty} b_m \sin(2\pi m \tau) \quad \tau \in [0,1] \quad (3.49)$$

and

$$R_{ee}(\tau) = a_o + \sum_{n=1}^{\infty} a_n \cos(2\pi n \tau) - \sum_{m=1}^{\infty} b_m \sin(2\pi m \tau) \quad \tau \in [-1,0].$$

Alternatively, the autocorrelation may be defined over $\tau \in [-1,1]$ by

$$R_{ee}(\tau) = a_o + \sum_{n=1}^{\infty} a_n \cos(2\pi n \tau) + \sum_{m=1}^{\infty} b_m \sin(2\pi m |\tau|) \quad (3.50)$$

Denote the basis functions in the integrand of Equation (3.46) as:

$$F(s,\tau,r,\rho) = \phi_j(s)\, \phi_j(\tau)\, \phi_i(r)\, \phi_i(\rho) \quad . \quad (3.51)$$

Then substitution of Equations (3.50) and (3.51) into Equation (3.46) gives:

$$E\{P^2\} = \int_0^1 \int_0^1 \int_0^r \int_0^\rho [a_o + \sum_{n=1}^{\infty} a_n \cos(2\pi n(s-\tau)) + \sum_{m=1}^{\infty} b_m \sin(2\pi m(s-\tau))] F(s,\tau,r,\rho)\,ds\,d\tau\,dr\,d\rho$$

$$= a_o \int_0^1 \int_0^1 \int_0^r \int_0^\rho F(s,\tau,r,\rho)\,ds\,d\tau\,dr\,d\rho +$$

$$+ \sum_{n=1}^{\infty} a_n \int_0^1 \int_0^1 \int_0^r \int_0^\rho \cos(2\pi n(s-\tau))F(s,\tau,r,\rho)\,ds\,d\tau\,dr\,d\rho +$$

$$+ \sum_{m=1}^{\infty} b_m \int_0^1 \int_0^1 \int_0^r \int_0^\rho \sin(2\pi m(s-\tau))F(s,\tau,r,\rho)\,ds\,d\tau\,dr\,d\rho. \qquad (3.52)$$

Now the integrals can be calculated independent of the actual form of the autocorrelation function $R_{ee}(\tau)$. In order to integrate the term containing $\sin(2\pi m|s-\tau|)$, it is necessary to break the integral into those portions in which $s-\tau$ is positive and those in which $\tau-s$ is positive. This is done by:

$$\int_0^1 \int_0^1 \int_0^r \int_0^\rho F(|\tau-s|)\,ds\,d\tau\,dr\,d\rho$$

$$= \int_0^1 \int_0^\rho \int_0^r \int_0^\tau F(\tau-s)\,ds\,d\tau\,dr\,d\rho$$

$$+\int_0^1\int_0^\rho\int_0^r\int_\tau^\rho F(s-\tau)\,ds\,d\tau\,dr\,d\rho$$

$$+\int_0^1\int_\rho^1\int_0^\rho\int_0^\tau F(\tau-s)\,ds\,d\tau\,dr\,d\rho$$

$$+\int_0^1\int_\rho^1\int_0^\rho\int_\tau^\rho F(s-\tau)\,ds\,d\tau\,dr\,d\rho$$

$$+\int_0^1\int_\rho^1\int_\rho^r\int_0^\rho F(\tau-s)\,ds\,d\tau\,dr\,d\rho \quad . \qquad (3.53)$$

Orthogonality of the basis functions and the functions in the expansion of $R_{ee}(\tau)$ causes most of the terms in the summations to be zero. It can be shown that the expression for the variance reduces to:

$$E\{P^2\} = a_o\int_1^0\int_1^0\int_0^r\int_0^\rho F(s,\tau,r,\rho)ds\,d\tau\,dr\,d\rho \quad \text{if } i=j=0$$

$$= \sum_{n=j,(i+j),(i-j)} a_n\int_0^1\int_0^1\int_0^r\int_0^\rho \cos(2\pi n(s-\tau))F(s,\tau,r,\rho)\cdot$$

$$\cdot ds\,d\tau\,dr\,d\rho + \sum_{m=j,(i+j),(i-j)} b_n\int_0^1\int_0^1\int_0^r\int_0^\rho \sin(2\pi m|s-\tau|)\cdot$$

$$\cdot F(s,\tau,r,\rho)\,ds\,d\tau\,dr\,d\rho \quad , \text{ otherwise.} \qquad (3.54)$$

This indicates that if the predominate frequency of the autocorrelation function is more than twice the highest frequency of the highest basis function, the variance of the error matrix will be small. As an example, consider a scalar equation which is perturbed by a sinusoid of frequency $2\pi n$:

$$\dot{x}(t) = a(t)\, x(t) + \cos\,(2\pi nt)\, x(t) \quad . \tag{3.55}$$

Use a basis set which is

$$\{1,\ \sin \pi t,\ \cos \pi t\} \quad . \tag{3.56}$$

A typical term in the error matrix will be of the form

$$e_{ij} = \int_0^1 \int_0^t \cos\,(n\pi s)\, \phi_j(s)\, ds\, \phi_i(t)\, dt \ ;\ i,j=1,2,3 \ . \tag{3.57}$$

For $\phi_j(s) = 1$ we get

$$e_{i1} = \int_0^1 \frac{\sin\,(n\pi t)}{n\pi}\, \phi_i(t)\, dt \tag{3.58}$$

and if $n > 1$, this term is identically zero for each of the basis functions. For $\phi_j(s) = \sin \pi t$ the error term is

$$e_{i2} = \int_0^1 - \frac{\cos\,((n+1)\pi t)}{(n+1)} - \left[\frac{\cos\,((n-1)\pi t)}{(n-1)}\right] \phi_i(t)\, dt. \tag{3.59}$$

If $n > 2$, this term is identically zero and the same condition ($n > 2$) makes the error for $\phi_j(t) = \cos \pi t$ equal zero. The reader should note that in analysis for the general case, the frequency of the autocorrelation function was important. In the example, the frequency of the time function was important. The connection is that the predominant frequency of the autocorrelation function is the predominant frequency of the time function. In fact, for the example considered, the time function and the autocorrelation function are identical.

A relation between the error "E" and the error induced in the approximation due to the perturbation "e" may be found by the following:

$$PA = f$$

$$(P - E)\,\tilde{A} = f$$

$$A - \tilde{A} = P^{-1}f - \tilde{A}$$

$$= P^{-1}(P-E)\,A - P^{-1}\,PA \qquad (3.60)$$

and finally

$$A - \tilde{A} = P^{-1}\,E\tilde{A} \quad .$$

Taking norms gives:

$$\frac{\|A - \tilde{A}\|}{\|\tilde{A}\|} \leq \|P^{-1}\| \cdot \|E\| \quad .$$

This relates the error in the perturbed approximation directly to the magnitude of the error matrix "E". The expected value of the error "E" is zero and we have shown that with adequate separation the error will not deviate much from zero. Thus, the perturbed approximate value will not deviate much from the correct approximate value.

3.2 Noise Sensitivity for Commutative Method

For the commutative algorithm, we are concerned with how much the function

$$\exp\left\{\int_0^T \Omega(s)\, ds\right\}$$

will change due to a noisy measurement of $\Omega(s)$. In order to investigate the sensitivity of this algorithm, expand the matrix function in a Taylor series and assume that "T" is small enough that the terms higher than the linear one may be neglected.

$$\exp\left\{\int_0^T \Omega(s)\, ds\right\} \triangleq \Phi(T) = I + \int_0^T \Omega(s)\, ds \quad . \tag{3.61}$$

Again let the measured value of $\Omega(s)$ be given by

$$\bar{\Omega}(s) = \Omega(s) + e(s) \tag{3.62}$$

where e(s) is a noise term. It is clear the error in the exponential will then be given by

$$\delta\, \Phi(T) = \int_0^T e(s)\, ds \quad . \tag{3.63}$$

Let e(s) have the following statistics:

$$\begin{aligned} E\{e(s)\} &= 0 \\ E\{e(s)\}e(s+\tau) &= R_{ee}(\tau) \quad . \end{aligned} \tag{3.64}$$

Then

$$E\{\delta\, \Phi(T)\} = 0 \quad .$$

In order to evaluate Var $\{\delta\, \Phi(T)\}$ expand $R_{ee}(z)$ in a cosine series on $\tau \in [-T,T]$. That is,

$$R_{ee}(\tau) = b_o + \sum_{n=1}^{\infty} \cos(n\omega z) \quad , \text{ for } z \in [-T,T] \tag{3.65}$$

where $\omega = \pi/T$. The variance is given by:

$$\text{Var}\{\delta\,\Phi(T)\} = \int_0^T \int_0^T R_{ee}(s-\tau)\,ds\,d\tau \quad . \tag{3.66}$$

Substitution of Equation (3.65) into (3.66) gives

$$\text{Var}\{\delta\,\Phi(T)\} = b_0 T^2 + \sum_{n=1}^{\infty} b_n \int_0^T \int_0^T \cos(n\omega(s-\tau))\,ds\,dz \quad . \tag{3.67}$$

Integration of Equation (3.67) gives:

$$\text{Var}\{\delta\,\Phi(T)\} = b_0 T^2 + \sum_{j=1}^{\infty} \frac{4\,b_{2j-1}}{(2j-1)\omega^2} \quad . \tag{3.68}$$

As before the algorithm proves to be insensitive to high frequency noise.

In particular, for the example case considered for the Galerkin technique (a sinusoid of period T, see Equation (3.55), b_0 will be zero. In addition, only one of the remaining b_i's will be non-zero and its subscript will be even. Therefore, for that case

$$\text{Var}\{\delta\,\Phi(T)\} = 0$$

and

$$\delta\,\Phi(T) = 0 \quad .$$

For the commutative approximation, it is clear that as the integration interval "T" gets large, the technique becomes less and less sensitive to noise. This is also true for the Galerkin technique but there the fact is hidden in

the analysis. For the Galerkin the frequency of the basis functions becomes lower as the integration interval is lengthened. This increases the frequency separation between the basis functions and the noise thereby reduces the algorithms sensitivity.

CHAPTER IV

APT APPLICATIONS

A test of the validity of the analysis is to compare the performance of systems using these different algorithms. The APT system currently under study by the AFWL at Kirtland Air Force Base uses a strapped-down system. A digital simulation program called OSUAPT has been developed (30). This program provides a convenient means for comparing the effects of the strapped-down system propagation algorithm on overall system performance.

As a point of comparison, runs were made using the OSUAPT in which the crude Euler technique now implemented in the APT system and the Galerkin outlined here were used to propagate the direction cosines. In order to simulate the effect of noise on the accuracy of the solutions, a zero mean white noise sequence of standard deviation σ was added to the angular rate measurements. Several runs were performed with different values for the standard deviation and using different time steps (update frequency). The system variables of interest are the tracking errors. The results of these runs are presented in Tables II and III.

TABLE II

TRACKING ACCURACY COMPARISON BETWEEN
GALERKIN AND EULER ALGORITHMS
USING .005 SECOND TIME STEP

Galerkin			Euler		
σ	RMS Tracking Error		σ	RMS Tracking Error	
	AZ	EL		AZ	EL
0	4.4988	3.0240	0	4.5052	3.3071
5	4.9451	4.3271	5	4.8422	4.5912
25	14.3249	15.4741	25	14.2317	15.5567

TABLE III

TRACKING ACCURACY COMPARISON BETWEEN
GALERKIN AND EULER ALGORITHMS
USING .01 SECOND TIME STEP

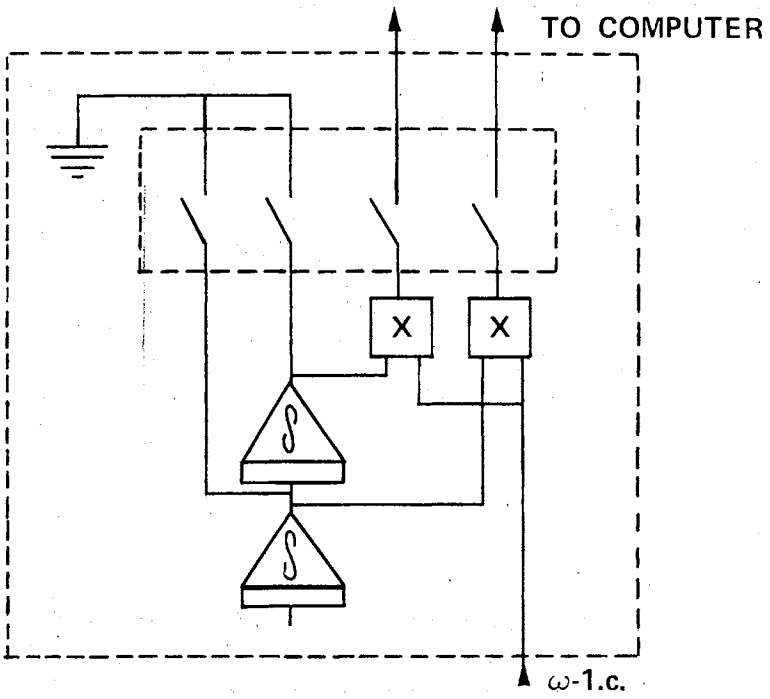
Galerkin			Euler		
σ	RMS Tracking Error		σ	RMS Tracking Error	
	AZ	EL		AZ	EL
0	4.5649	3.0061	0	4.7473	3.7165
5	5.0376	4.3071	5	5.6550	4.7425
25	14.3680	15.6716	25	14.1381	16.1867

The Galerkin Algorithm used for Tables II and III to propagate the direction cosines is a modified second-order approach. In the modified second-order algorithm, the coefficient on the constant is taken to be the initial condition and the coefficient on the linear term is allowed to vary. The resulting system of linear equations is of the same order as a first-order method. An error plot for this algorithm would approach that for the second-order algorithm shown in Figure 16. If the second-order algorithm could be applied with the same speed as the first-order, it would still be slower for the same accuracy than the commutative algorithm. APT results using a commutative algorithm will be similar to those using Galerkin with perhaps slight improvement. In light of the commutative algorithm's speed (smaller number of computations), it is probably the most satisfactory for the APT application.

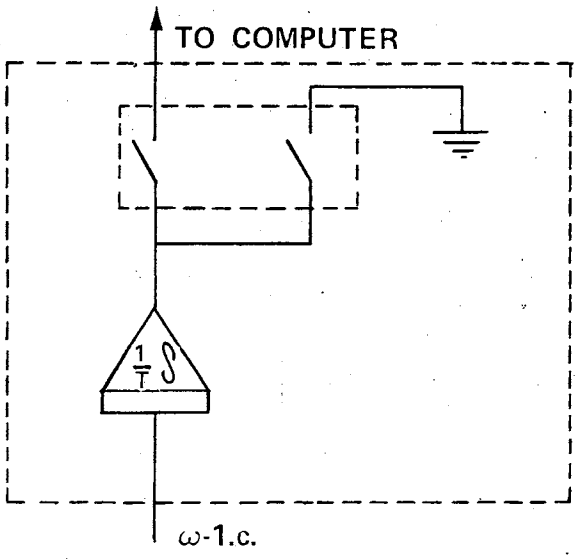
The results in Tables II and III show that generally a more sophisticated integration algorithm can give some improvement in tracking accuracy, but not a tremendous amount. The major reason for this is the fact that the angular rate measurements around the "y" and "z" axes are not available continuously, but are taken as the sampled and held commanded rates. The sampling rate is not significantly faster than the cosine matrix update rate. The Galerkin algorithm

then does not have a good chance to "average out" the noise since it appears as a constant bias over all or most of the update interval. This severely handicaps the technique. The "x" axis (roll channel) provides a chance for some improvement since its output is continuous and the noise effects there may be averaged. It is interesting to note that the increase in error for the Galerkin technique as it is shifted from a .005 to .01 second time step is much smaller than that for the Euler. This suggests that although the sampling rate for the angular rates sets an upper bound on propagation accuracy, it is possible to maintain that accuracy using a more sophisticated algorithm at a slower rate, thereby reducing computer load.

In order to implement either the Galerkin algorithm or the commutative algorithm, it is necessary to make some system modifications to generate the weighted integrals. Figures 20 and 21 outline the necessary modifications.



a.) Averager for Variational Method



b.) Averager for STM Method

Figure 21. Schematic of Averagers

CHAPTER V

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

1. Results

During the course of this investigation, several things have been accomplished:

1. A theoretical basis for the commutative algorithms is given.
2. Galerkin's technique was applied to the propagation of direction cosines.
3. An analytical error analysis was performed for a modified second-order Galerkin method and an Euler method.
4. A convergence proof for the commutative algorithms was derived and one for the Galerkin technique was shown.
5. An exact analytical solution for a general coning motion was derived.
6. Convergence with increasing update frequency or approximation order of the commutative algorithm

and the Galerkin technique were demonstrated computationally.

7. Time requirements for the Galerkin, state transition matrix, Euler, and quaternion algorithms were determined for a typical airborne computer.
8. Sensitivity of the Galerkin algorithm and the commutative algorithm to imperfect knowledge of the coefficient matrix (t) was found.
9. The modified second-order Galerkin method was used to update the cosines in the APT simulation.

2. Conclusions

On the basis of the studies mentioned, it can be concluded that:

1. The crude Euler algorithm is not stable and is an inferior method for updating the direction cosine matrix. This situation is worsened by the presence of noise on the angular rate signals. A more reasonable approach is to expand the matrix exponential of the commutative approximation in a Taylor series and truncate to two terms. This results in no increase in computer load. The form of the solution obtained is similar, but this algorithm is much less sensitive to noise.

2. The Galerkin algorithm is a reasonable method, but the commutative algorithms are more efficient for maintaining an analytic platform. The convergence proofs for these two methods give some insight into the reason for this. The commutative algorithm gives the correct solution form and needs only to be applied several times in order to make the approximation error arbitrarily small. Each additional application of the technique increases the computer time by a fixed amount. The Galerkin approximation, on the other hand, needs to have a sufficiently large number of basis functions in order to make the approximation error arbitrarily small. Each additional basis function, however, increases the computer time by n^2 where "n" is the approximation order. In addition, the first-order Galerkin approximation requires approximately the same amount of time as a single iteration of the commutative algorithm.

3. The accuracy of the APT tracking system can be improved by using a better propagation algorithm. There is evidence which indicates that it is not desirable to propagate the direction cosines exactly. The high frequency portion of the exact

orientation vector causes excessive transients in the pointing system. It is desirable to follow only the low frequency portion of the solution. If the high frequency excursions can be considered commutative, then they may be averaged out with no loss of accuracy. The commutative algorithm with a suitable time step will smooth the direction cosine calculations in roughly this manner. The algorithm should be updated at the lower frequency (.01 second time step) as indicated by Tables II and III. At this rate the algorithm with machine evaluations of the transcendental function will take less time than the routine presently implemented in the APT system. It may be satisfactory to approximate the transcendental functions by a two term expansion as indicated in Section 2. (See Equation 2.19.) Use of this in the .01 second loop would require half the time now used by the propagation algorithm.

3. Recommendations

1. A commutative algorithm should be used in the APT system.

2. As update frequency is increased, the commutative algorithms show a reduction in error due to approximation, but an increase in error due to noise in the $\Omega(t)$ matrix. As update frequency is decreased, the converse is true. A study should be undertaken to determine what factors affect these errors and given the details of a mission, what is the optimal update frequency.

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APPENDIX A

CORRESPONDENCE BETWEEN THE EULER INTEGRATION TECHNIQUE AND A FIRST-ORDER GALERKIN TECHNIQUE

For the variational formulation assume a vector differential equation of the form:

$$\dot{x} - g(x,t) = 0 \ . \tag{A.1}$$

Then this may be satisfactorily approximated over the interval $t \in [t_a,t_b]$ by:

$$\hat{x}(t) = x(t_a) + a \cdot (t - t_a) \tag{A.2}$$

where a is chosen to minimize

$$J(x) = \int_{t_a}^{t_b} [\dot{\hat{x}} - g(x,t)] \, dt \tag{A.3}$$

or

$$\begin{aligned} \delta J(x) &= \delta \int_{t_a}^{t_b} [\dot{\hat{x}} - g(x,t)] \, dt \\ &= \int_{t_a}^{t_b} [a - g(x(t_a) + a \cdot (t - t_a),t)] \, (t - t_a)dt. \\ &= 0 \end{aligned} \tag{A.4}$$

Let $z = t-t_a$, $t = z+t_a$, and $dt = dz$, then:

$$\delta J(x) = \int_0^{t_b-t_a} [a - g(x(t_a)+a \cdot z,\ z+t_a)]\ z\ dz$$

and

$$0 = \frac{a(t_b - t_a)^2}{2} - \int_0^{t_b-t_a} g(x(t_a)+a\cdot z,\ z+t_a) z\ dz \tag{A.5}$$

From the mean value theorem there exists some t_c, t_a t_c t_b such that

$$\frac{a(t_b-t_a)^2}{2} = g(x(t_a)+a\cdot t_c,\ t_c+t_a)\ t_c(t_b - t_a) \tag{A.6}$$

but t_c can be given by

$$t_c = \alpha(t_b - t_a)+t_a \qquad 0 \leq \alpha \leq 1$$

and

$$\frac{a(t_b - t_a)^2}{2} = g(x(t_a)+a\cdot\alpha(t_b-t_a),\ \alpha(t_b-t_a) + t_a)\cdot\alpha(t_b-t_a)(t_b-t_a) .$$

This gives:

$$a = g(x(t_a)+a\cdot\alpha(t_b-t_a),\ \alpha(t_b-t_a)+t_a)(2\alpha) . \tag{A.7}$$

It can be shown that:

$$\lim_{t_b \to t_a} \alpha = 1/2 .$$

It follows that

$$\lim_{t_b \to t_a} a = g(x(t_a),t_a) .$$

Substituting this value for a into the assumed solution form gives:

$$\hat{\hat{x}}(t) = x(t_a) + g(x(t_a), t_a)\ (t - t_a) \qquad (A.8)$$

but

$$g(x(t_a), t_a) = \dot{x}(t_a)$$

so

$$\hat{x}(t) = x(t_a) + \dot{x}(t_a)\ (t - t_a) \qquad (A.9)$$

which is exactly the Euler integration formula.

APPENDIX B

A MODAL MATRIX METHOD FOR DECOUPLING THE DIRECTION COSINE EQUATIONS

It can be shown that there exists a matrix T such that

$$T^{-1}\Lambda T = \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) \\ -\omega_3(t) & 0 & \omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{bmatrix} = \Omega(t) \qquad (B.1)$$

where Λ is the diagonal matrix. For the skew symmetric matrix $\Omega(t)$, it can be shown that

$$\Lambda = \begin{bmatrix} j\lambda & 0 & 0 \\ 0 & -j\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (B.2)$$

where $\pm j\lambda$, 0 are the eigen values of $\Omega(t)$ and

$$\lambda = \sqrt{\omega_1^2(t) + \omega_2^2(t) + \omega_3^2(t)} \quad . \qquad (B.3)$$

Rearrange the first equation into the form

$$\Lambda T = T\Omega(t) \quad . \qquad (B.4)$$

Considering the last row of Equation (B.4),

$$[0,0,0]T = [t_{31}\ t_{32}\ t_{33}] \quad \Omega(t) \quad , \qquad (B.5)$$

one obtains the three relations:

$$t_{31} = \omega_1(t),\ t_{32} = \omega_2(t),\ t_{33} = \omega_3(t) \quad . \tag{B.6}$$

The direction cosine equation for a single column vector of the cosine matrix is:

$$\dot{C} = \Omega(t)\ C$$

where

$$C^T = [C_1(t)\ C_2(t)\ C_3(t)] \quad .$$

In terms of Equation (B.1)

$$\dot{C} = T^{-1} \Lambda\ TC$$

$$T\dot{C} = \Lambda TC \quad . \tag{B.7}$$

Consider the last element of the vector on either side of the equation.

$$\omega_1(t)\dot{C}_1(t) + \omega_2(t)\dot{C}_2(t) + \omega_3(t)\dot{C}_3(t) = 0$$

or

$$\dot{C}_3(t) = \begin{bmatrix} -\dfrac{\omega_1(t)}{\omega_3(t)} & -\dfrac{\omega_2(t)}{\omega_3(t)} \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} \quad . \tag{B.8}$$

The differential equations for C_1 and C_2 may be written as

$$\begin{bmatrix} \dot{C}_1(t) \\ \dot{C}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \omega_3(t) \\ -\omega_3(t) & 0 \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} + \begin{bmatrix} -\omega_2(t) \\ \omega_1(t) \end{bmatrix} C_3(t) \quad . \tag{B.9}$$

The state transition matrix is given by:

$$\Phi(t_b,t_a) = \begin{matrix} \cos\ \Omega(t_a,t_b) & \sin\ \Omega(t_a,t_b) \\ -\sin\ (t_a,t_b) & \cos\ \Omega(t_a,t_b) \end{matrix} \quad , \tag{B.10}$$

where

$$\Omega(t_a,t_b) = \int_{t_a}^{t_b} \omega_3(t)\, dt \quad .$$

Then

$$\begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \Phi(t,t_a) \begin{bmatrix} C_1(t_a) \\ C_2(t_a) \end{bmatrix} + \int_{t_a}^{t} \Phi(t,\tau) \begin{bmatrix} -\omega_2(\tau) \\ \omega_1(\tau) \end{bmatrix} C_3(\tau)\, d\tau \quad , \qquad (B.11)$$

and

$$\begin{bmatrix} \dot{C}_1(t) \\ \dot{C}_2(t) \end{bmatrix} = \dot{\Phi}(t,t_a) \begin{bmatrix} C_1(t_a) \\ C_2(t_a) \end{bmatrix} + \begin{bmatrix} -\omega_2(t) \\ \omega_1(t) \end{bmatrix} C_3 + \int_{t_a}^{t} \dot{\Phi}(t,\tau) \begin{bmatrix} -\omega_2(\tau) \\ \omega_1(\tau) \end{bmatrix} C_3(\tau)\, d\tau \quad ,$$

where

$$\dot{\Phi}(t,t_a) = \begin{bmatrix} -\sin \Omega(t_a,t) & \cos \Omega(t_a,t) \\ -\cos \Omega(t_a,t) & -\sin (t_a,t) \end{bmatrix} \omega_3(t)\, \omega_3(t) \quad .$$

Then

$$C_3(t) = \begin{bmatrix} \frac{-\omega_1(t)}{\omega_3(t)} & \frac{-\omega_2(t)}{\omega_3(t)} \end{bmatrix} \{ \dot{\Phi}(t,t_a) \begin{bmatrix} C_1(t_a) \\ C_2(t_a) \end{bmatrix} + \begin{bmatrix} -\omega_2(t) \\ \omega_1(t) \end{bmatrix} C_3(t) + \int_{t_a}^{t} \dot{\Phi}(t,\tau) \begin{bmatrix} -\omega_2(\tau) \\ \omega_1(\tau) \end{bmatrix} C_3(\tau)\, d\tau \} \quad . \qquad (B.12)$$

The result is a third order scalar integro-differential equation. The solution to this is not trivial, but may lend itself to some approximation.

APPENDIX C

CORRESPONDENCE OF RITZ'S TECHNIQUE AND GALERKIN'S TECHNIQUE

The basic problem in variational calculus is to minimize an integral of the form:

$$I(u) = \int_R F(x,u,u_x)\, dx \quad . \tag{C.1}$$

In the integrand, u is a function of the independent variable(s) x and u_x denotes partial differentiation of u with respect to the independent variable(s). The integral is over some region "R" with coordinates "x" and at the boundary of the region "Γ" the boundary conditions are given by

$$u = \varphi(s) \text{ on } \Gamma \quad . \tag{C.2}$$

In particular, if the region "R" is a two dimensional cartesian space D, then

$$I(u) = \iint_D F(x,y,u,u_x,u_y)\, dx\, dy \tag{C.3}$$

$$u = \varphi(s) \text{ on } \Gamma \quad .$$

In some cases minimization of I(u) may be achieved by application of the exact techniques of variational calculus (e.g. solution of the associated Euler Lagrange equation).

The function u which minimizes I(u) is denoted by u^*. In some cases it is not possible to find u^* exactly and some approximation technique must be used. Ritz's method proceeds in the following manner. Consider a family of functions depending on several parameters

$$u = \psi(x,y,a_1,a_2,\ldots,a_n) \quad . \tag{C.4}$$

If this function is substituted into the functional I(u), the functional will become a function of the variables $a_1,a_2,\ldots,a_n$. Minimization of this function may be carried out by the techniques of ordinary calculus:

$$\frac{\partial I}{\partial a_k} = 0 \qquad k = 1,2,\ldots,n \quad . \tag{C.5}$$

By solving these equations we obtain values for the a_k which will give a function $u(x,y,a_1,a_2,\ldots,a_n)$ which minimizes the functional I(u). Denote the a_k which give a minimum as $\bar{a}_k$ and the resultant u as $\bar{u}$. It should be pointed out that $\bar{u}$ will not necessarily give the smallest possible value of I(u) since we have constrained the form of the function u by Equation (C.4). This has the effect of reducing the set of functions over which the minimization is applicable. As more a_k are included, the family of functions over which the minimization is done becomes larger. Each successive family contains all of the preceding ones. Denote u_i to be the family depending on i free parameters

$$u_i = \psi(x,y,a_1,a_2,\ldots,a_i) \quad . \tag{C.6}$$

Then

$$I(\bar{u}) \geq I(\bar{u}_2) \geq \ldots \quad . \tag{C.7}$$

The successive $I(\bar{u}_i)$ are non-increasing but we would like to show that

$$\lim_{n\to\infty} I(u_n) = I(u^*)$$

where u^* gives $I(u^*)$ an absolute minimum value. In order for this to be true, it is necessary that the family of functions u_i be complete. The family is complete if for any continuous function u with continuous partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ and any positive number $\varepsilon > 0$, one can find an integer n and function u_n^* from the n^{th} family

$$u_n^* = \phi_n(x,y,a_1^*,a_2^*,\ldots,a_n^*) \tag{C.8}$$

such that everywhere in the region D

$$|u_n^* - u| < \varepsilon, \quad \left|\frac{\partial u_n^*}{\partial x} - \frac{\partial u}{\partial x}\right| < \varepsilon, \quad \left|\frac{\partial u_n^*}{\partial y} - \frac{\partial u}{\partial y}\right| < \varepsilon \quad . \tag{C.9}$$

In other words, if the family ϕ is complete, then any admissable function and its partial derivatives may be approximated arbitrarily close by some function ϕ_n. It can be shown that $\phi_k = \sin(k\pi x)$ and $\phi_k = x^k(1-x)$ are complete (23). Since u_n^* can be arbitrarily close to u^* and since $F(x,y,u,u_x,u_y)$ is continuous, then the difference in the two functionals can be made arbitrarily small, i.e.,

$$I(u_n^*) - I(u^*) = \iint_D [F(x,y,u_n^*, \frac{\partial u_n}{\partial x}, \frac{\partial u_n^*}{\partial y}) - F(x,y,u^*, \frac{\partial u^*}{\partial x}, \frac{\partial u^*}{\partial y})]dx\,dy < \varepsilon'. \tag{C.10}$$

Since $\bar{u}_n$ gives I a minimum value over the set of functions

$$u_n = \phi_n(x,y,a_1,a_2,\ldots,a_n) \quad ,$$

then $I(\bar{u}_n) \leq I(u_n^*)$ and finally,

$$I(u^*) \leq I(\bar{u}_n) \leq I(u_n^*) \leq I(u^*) + \varepsilon' \quad . \tag{C.11}$$

Galerkin's technique can be stated quite easily. Suppose we are given the equation:

$$L(u) = 0 \tag{C.12}$$

where L is a differential or an integral operator in one dimension. (The method applies directly to higher dimensions.) Then we shall seek an approximate solution of the form:

$$\bar{u}_n(x) = \sum_{i=\phi}^{n} c_i \phi_i(x) \tag{C.13}$$

where the ϕ_i are a system of functions chosen beforehand satisfying the boundary conditions. Find the c_i by solving

$$\int L(\bar{u}(x))\, \phi_j(x)\, dx = \int L\left(\sum_{i=0}^{n} c_i \phi_i(x)\right) \phi_j(x)\, dx = 0$$

$$(j = 0,1,2,\ldots,n) \quad . \tag{C.14}$$

Consider the one dimensional problem of the calculus of variations

$$I(u) = \int_a^b F(x,u,u_x)\, dx \tag{C.15}$$

with

$$u(a) = u(b) = 0 \tag{C.16}$$

(homogeneous boundary conditions do not lessen the generality here since the case where they are not may be transformed to a problem where they are by a change of variables). Approximate the solution to this equation by Ritz method. Let

$$u_n = \sum_{i=0}^{n} a_i \phi_i(x) \tag{C.17}$$

then

$$I(u_n) = \int_a^b F(x,u_n,u_n')\, dx \tag{C.18}$$

and this is minimized by:

$$\frac{\partial I(u_n)}{\partial a_i} = 0 = \int_a^b \left[\frac{\partial F}{\partial u_n} \frac{\partial u_r}{\partial a_i} + \frac{\partial F}{\partial u_n} \frac{\partial u_n}{\partial a_i} \right] dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial u_n} \phi_n + \frac{\partial F}{\partial u_n} \phi_i \right] dx \quad . \tag{C.19}$$

Integration by parts gives:

$$\int_a^b \frac{\partial F}{\partial u_n} \phi_i = \frac{\partial F}{\partial u'} \phi_i \Big|_a^b - \int_a^b \frac{d}{dx} \frac{\partial F}{\partial u_n'} \phi_i \, dx \quad . \tag{C.20}$$

Recognition of the homogeneous boundary conditions and substitution gives:

$$\frac{\partial I(u_n)}{\partial a_i} = \int_a^b \left[\frac{\partial F}{\partial u_n} - \frac{d}{dx} \frac{\partial F}{\partial u_n'} \right] \phi_i dx = 0 \quad (i = 0,1,\ldots,n). \tag{C.21}$$

Now it is necessary to recognize that the Euler-Lagrange equation is the defining equation for u, i.e., along the trajectory of u

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} = 0 \quad . \tag{C.22}$$

If we define an operator L by

$$L(u) = \frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} \tag{C.23}$$

and apply Galerkin's method using

$$u_n = \sum_{i=0}^{n} a_i \phi(x) \quad , \tag{C.24}$$

we get:

$$\int_a^b \left[\frac{\partial F}{\partial u_n} - \frac{d}{dx} \frac{\partial F}{\partial u_n'} \right] \phi_i \, dx = 0 \quad (i=0,1,\ldots,n) \; ; \tag{C.25}$$

then it is obvious that the Ritz technique and the Galerkin technique are equivalent. One important difference should be noted, however. Ritz's technique requires a variational formulation of the problem. An equation of motion is then found and the approximation proceeds. Galerkin's technique,

on the other hand, proceeds directly from the equation of motion and does not require a variational formulation, and is, therefore, more generally applicable.

APPENDIX D

CONVERGENCE OF THE COMMUTATIVE APPROXIMATION AND GALERKIN'S TECHNIQUE

The conditions of this proof are derived by Kantorovich (20). Consider a linear functional equation of the form

$$Kx = y \tag{D.1}$$

where $x, y \in X$, X is a normed linear space, and $K:X \to X$. Consider also an approximate equation

$$\overline{K}\,\overline{x} = \overline{y} \tag{D.2}$$

where $\overline{x}, \overline{y} \in \overline{X}$, $\overline{X}$ is a normed linear space, and $\overline{K}:\overline{X} \to \overline{X}$. The space $\overline{X}$ is chosen to be more simple than the space X and the operator $\overline{K}$ is supposed to be near K. For this problem, the operators may be defined by:

$$K = I - \lambda H$$

and

$$\overline{K} = \overline{I} - \lambda \overline{H}$$

where I and $\overline{I}$ are the identity maps on X and $\overline{X}$, respectively, $\overline{H}:X \to X$, and $H:\overline{X} \to \overline{X}$. The space $\overline{X}$ is complete and is isomorphic with X' where $X' \subset X$. Let $\phi_0:X' \to \overline{X}$ and be invertible, i.e.,

$\phi_o^{-1}\,\phi_o = I$ and $\phi_o\phi_o^{-1} = I$. Further consider that an extended ϕ exists where $\phi: X \to \overline{X}$ and $\phi x' = \phi_o x' \;\forall\; x' \in X'$. Investigate the equations

$$K x = x - \lambda H x = y \quad ,$$

and

$$\overline{K}\,\overline{x} = \overline{x} - \lambda \overline{H}\,\overline{x} = \overline{y} = \phi y \quad .$$

The first condition is that the operators be close, or:

$$\| \phi\, Hx' - \overline{H}\, \phi_o x' \| \leq \epsilon \; \|x'\| \;, \; x' \in X' \tag{D.3}$$

or equivalently that:

$$\| \phi\, Kx' - \overline{K}\, \phi x' \| \leq \epsilon \; |\lambda| \; \|x'\| \;, \; x' \in X' \quad .$$

The second condition is that it be possible to approximate an element of the form Hx by an element x' or that $\forall$ x $\in$ X, $\exists$ x' $\in$ X' $\ni$

$$\|Hx - x'\| \leq \epsilon_1 \; \|x\| \quad . \tag{D.4}$$

The third condition is that it be possible to approximate y by an element y' or that $\forall$ y $\in$ X, $\exists$ y' $\in$ X' $\ni$

$$\|y - y'\| \leq \epsilon_2 \; \|y\| \quad . \tag{D.5}$$

The following theorem is stated without proof. (For proof, see Kantorovich (22), page 43.)

Theorem. If:

(1) The operator K has an inverse ($\|K^{-1}\| < t$);

(2) The operator $\overline{K}$ is such that the existence of a solution to Equation (D.2) for any $\overline{y}$ implies the uniqueness of that solution (that the

operator $\bar{H}$ is completely continuous is sufficient);

(3) $$\lim_{n\to\infty} \epsilon \|\phi\| \|\phi_o^{-1}\|^2 = 0 \qquad \text{(D.6)}$$

$$\lim_{n\to\infty} \epsilon_1 \|\phi\|^2 \|\phi_o^{-1}\|^2 = 0 \qquad \text{(D.7)}$$

$$\lim_{n\to\infty} \epsilon_2 \|\phi\|^2 \|\phi_o^{-1}\|^2 = 0 \quad ; \qquad \text{(D.8)}$$

then the approximate equations are solvable (beginning with a certain n), and the convergence of the approximate solutions to the exact occurs:

$$\lim_{n\to\infty} \|x^* - \phi_o^{-1} \bar{x}_o\| = 0$$

where $\bar{x}_o$ is the solution to Equation (D.2).

First, examine the commutative approximation. The equation we desire a solution to is:

$$\dot{x} = A(t)\, x(t) + z(t) \quad , \; t \in [a,b] \quad . \qquad \text{(D.9)}$$

Rewrite this as a Volterra equation

$$x(t) - \int_a^t A(s)\, x(s)\, ds = x(a) + \int_a^t z(t)\, dt \quad . \qquad \text{(D.10)}$$

Make the definition that

$$y(t) \triangleq x(a) + \int_a^t z(t)\, dt \quad . \qquad \text{(D.11)}$$

Now the equation can be in the form of Equation (D.1) with

$$Hx \triangleq \int_a^t A(s)\, x(s)\, ds \quad . \qquad \text{(D.12)}$$

In order to establish that the integral equation, Equation (D.10), has an inverse which is bounded, consider

$$x(t) = \int_0^t A(s)\, x(s)\, ds + y(t) \quad , \; t \in [0,T] \quad . \tag{D.13}$$

Substitute successively for x(s) in the integral to obtain the Neumann series:

$$\begin{aligned} x(t) = y(t) + & \int_0^t A(s)\, y(s)\, ds + \\ & + \int_0^t A(s) \int_0^s A(s_1)\, y(s_1)\, ds_1\, ds + \ldots \\ & + \int_0^t A(s) \int_0^s A(s_1) \ldots \\ \ldots & \int_0^{s_{n-1}} A(s_n) y(s_n) ds_n ds_{n-1} \ldots ds_1 ds + \ldots \end{aligned} \tag{D.14}$$

For this problem x is a member of the space of continuous functions or X = C[a,b]. The approximate equation due to the commutative approximation is

$$\bar{x}(t) - \int_a^t \bar{A}(s)\, \bar{x}(s)\, ds = \bar{y}(t) \quad . \tag{D.15}$$

The functions x(t) and y(t) are continuous vector functions and the function A(s) is a continuous matrix function of y(t), $x(t) \in C[0,T]$ ($C^n[0,T]$ denotes the n^{th} Cartesian product of space C[0,T] with itself) and $A(t) \in C^{nxn}[0,T]$

($C^{nxn}[0,T]$ denotes the Cartesian product of $C^n[0,T]$ with itself).

$$\|y(t)\|_\infty^n = \max_i \|y_i(t)\|_\infty \tag{D.16}$$

where $y_i(t)$ is the i^{th} element of the vector $y(t)$ and $\|\cdot\|_\infty$ denotes the uniform norm on $[0,T]$. With this definition for a vector norm, the matrix norm becomes

$$\|A(t)\|_\infty^{nxn} = \max_i \sum_j \|a_{ij}(t)\|_\infty \tag{D.17}$$

where $a_{ij}(t)$ is the ij^{th} element of the matrix $A(t)$. Let

$$M = \|A(t)\|_\infty^{nxn}$$

and

$$U = \|y(t)\|_\infty^n \tag{D.18}$$

then by taking norms of the infinite series (D.14), it is seen that

$$\begin{aligned}\|x(t)\|_\infty^n \le U + \int_0^t MU\,ds + \int_0^t M \int_0^s MU\,ds_1\,ds \\ + \int_0^t M \int_0^s M \cdots \int_0^{s_{n-1}} MU\,ds_n\,ds_{n-1}\cdots ds_1\,ds + \cdots \\ \le \left(1 + Mt + \frac{M^2t^2}{2} + \frac{M^3t^3}{3!} + \cdots\right) U \qquad (D.19) \\ \le \left(1 + MT + \frac{M^2T^2}{2} + \frac{M^3T^3}{3!} + \cdots\right) U \quad . \qquad (D.20)\end{aligned}$$

The series (D.14) converges uniformly and absolutely to $x(t) \in C^n [0,T]$. This means that K^{-1} exists. With the restriction that $M < \infty$ and $U < \infty$, $\|x(t)\|_\infty^n < +\infty$, since $x(t)$ may also be written as

$$x = K^{-1} y \tag{D.21}$$

where

$$Kx = x(t) - \int_0^t A(s)\, x(s)\, ds \quad .$$

It follows that

$$\|x\| = \|K^{-1} y\| \quad . \tag{D.22}$$

Recognize that

$$e^{MT} = 1 + MT + \frac{M^2T^2}{2} + \frac{M^3T^3}{3!} + \ldots \tag{D.23}$$

Combining (D.18), (D.20) and (D.21) gives

$$\frac{\|K^{-1}y\|}{\|y\|} \leq e^{MT} \quad . \tag{D.24}$$

This implies that:

$$\|K^{-1}\| \leq e^{MT} \quad . \tag{D.25}$$

Therefore, K^{-1} is bounded.

Equation (D.12) is also a map on the continuous functions and this means that $\bar{X} = X$. For this case,

$$\phi_o = \phi_o^{-1} = \phi = I \quad ,$$

and more important

$$\epsilon_1 \equiv 0 \qquad \text{and} \qquad \epsilon_2 \equiv 0 \quad .$$

The only problem is then to show that

$$\lim_{n\to\infty} \epsilon = 0$$

or that the approximate operator can be made arbitrarily close to the exact one. For the commutative approximation the interval [a,b] is divided into N equal intervals by placing divisions at each t_i in the sequence

$$a = t_0, t_1, t_2, \ldots, t_{N-1}, t_N = b$$

where $t_i - t_{i-1} = (b-a)/N$. The matrix valued function $\bar{A}(s)$ is a piecewise continuous approximation to A(s) defined by:

$$\begin{aligned}
\bar{A}(s) &= A(U_1)\ U_1 \in [t_0,t_1] \quad \forall\ s \in [t_0,t_1] \\
&\vdots \\
&= A(U_i)\ U_i \in [t_{i-1},t_1] \quad \forall\ s \in [t_{i-1},t_1] \\
&\vdots \\
&= A(U_N)\ U_N \in [t_{N-1},t_N] \quad \forall\ s \in [t_{N-1},t_N] \ . \qquad (D.26)
\end{aligned}$$

Condition (D.3) is that

$$\left\| \int_a^t A(s)\, x(s)\, ds - \int_a^t A(s)\, x(s)\, ds \right\|_\infty^n \le \epsilon \left\| x \right\|_\infty^n$$

$$\left\| \int_a^t A(s)\, x(s)\, ds - \int_a^t \bar{A}(s)\, x(s)\, ds \right\|_\infty^n \le$$

$$\le \int_a^t \left\| A(s) - \bar{A}(s) \right\|_\infty^{nxn} ds \quad \left\| x \right\|_\infty^n \ .$$

Let

$$\epsilon = \int_a^t \| A(s) - \bar{A}(s) \|_{\infty}^{nxn} ds \qquad (D.27)$$

then it is clear that

$$\lim_{n \to \infty} \epsilon = 0$$

if A(s) is continuous on [a,b], and thus convergence is assured.

Convergence of the commutative approximation may be demonstrated in a more straightforward manner. In order to approximate

$$x = K^{-1} y \quad ,$$

solve

$$x = \bar{K}^{-1} y$$

where $\bar{K}$ is the piecewise constant approximation. It is esaily established that $\bar{K}^{-1}$ exists and is bounded by argument parallel to that for K^{-1}. The norm of the error in making the approximation is given by:

$$\begin{aligned} \| x - \bar{x} \| &= \| (K^{-1} - \bar{K}^{-1}) \, y \| \\ &\leq \| K^{-1} - \bar{K}^{-1} \| \; \| y \| \quad . \end{aligned} \qquad (D.28)$$

The difference in the inverses can be expressed as:

$$\begin{aligned} K^{-1} - \bar{K}^{-1} &= (I - \bar{K}^{-1} K) \, K^{-1} \\ &= (I - \bar{K}^{-1} K) \, \bar{K}^{-1} + (I - \bar{K}^{-1} K)(K^{-1} - \bar{K}^{-1}) \\ &= \bar{K}^{-1} (\bar{K} - K) \, \bar{K}^{-1} + \bar{K}^{-1} (\bar{K} - K) \, (K^{-1} - \bar{K}^{-1}) \, . \end{aligned}$$

Taking norms gives:

$$\|K^{-1} - \bar{K}^{-1}\| \leq \|\bar{K}^{-1}\|^2 \|\bar{K} - K\| + \|\bar{K}^{-1}\| \cdot \|\bar{K} - K\| \|K^{-1} - \bar{K}^{-1}\| .$$

Rearrange this to yield:

$$\|K^{-1} - \bar{K}^{-1}\| \leq \frac{\|\bar{K}^{-1}\| \|\bar{K} - K\|}{1 - \|\bar{K}^{-1}\| \|\bar{K} - K\|} . \qquad (D.29)$$

This expression will only make sense if

$$\|\bar{K}^{-1}\|^{-1} > \|\bar{K} - K\|$$

which will be true in the limit since, as shown before,

$$\lim_{n\to\infty} \|\bar{K} - K\| = 0 . \qquad (D.30)$$

The approximation error is then bounded by:

$$\|x - \bar{x}\| \leq \frac{\|\bar{K}^{-1}\| \|\bar{K} - K\|}{1 - \|\bar{K}^{-1}\| \|\bar{K} - K\|} \|y\| . \qquad (D.31)$$

Then from (D.30) and (D.31) it follows that

$$\lim_{n\to\infty} \|x - \bar{x}\| = 0 .$$

Consider the application of Galerkin's technique to the Volterra equation, Equation (D.10). By this technique we seek an approximate solution of the form

$$x' = c_1\omega_1 + c_2\omega_2 + \ldots + c_n\omega_n \qquad (D.32)$$

where the ω_i form an orthogonal and normed system.

The coefficients c_i are selected by the condition of least mean square error.

Define

$$h_{ik} \triangleq \int_0^1 \int_0^s H(t)\ \omega_k(t)\ \omega_i(s)\ dt\ ds \qquad (D.33)$$

and

$$\eta_i \triangleq \int_0^1 y(s)\ \omega_i(s)\ ds \quad . \qquad (D.34)$$

The approximate equation is given by:

$$C_i - \sum_{k=1}^{n} h_{ik}\ C_k = \eta_i \qquad (i = 1,2,\ldots,n) \quad . \qquad (D.35)$$

Now assume that $X = L^2(a,b)$, $\bar{X} = R^n$, and X' is the set of functions of the form (D.32).

The mapping ϕ is defined by:

$$x = \phi x \ ; \ x = (C_1,C_2,\ldots,C_n) \ ; \ C_i = \int_0^1 x\ \omega_i\ ds$$

$$x' = \phi_o^{-1}\ x = \phi_o^{-1}\ [(C_1,C_2,\ldots,C_n)]$$

$$= C_1\omega_1 + C_2\omega_2 + \ldots + C_n\omega_n \quad .$$

Obviously,

$$\|\phi\| = \|\phi_o\| = \|\phi_o^{-1}\| = 1 \quad .$$

It can be easily demonstrated that

$$\phi\ Hx' = \bar{H}\ \phi\ x' \ ;$$

therefore, condition (D.3) is

$$\|\phi\ H\ x' - \bar{H}\ \phi\ x'\| = 0 \quad .$$

This gives:

$$\epsilon = 0 \quad .$$

Define $[H]_{n,n}$ to be the n^{th} partial sum of the double Fourier series of H(t), i.e.,

$$[H]_{n,n} = \sum \sum h_{ij}\, \omega_i(s)\, \omega_j(t) \quad . \tag{D.36}$$

Then set

$$x' = \int_0^s [H]_{n,n}\, x\, dt$$

$$\|Hx - x'\|_2 = \left\{ \int_0^1 \int_0^s [(H(t) - [H]_{n,n})\, x(t)\, dt]^2 ds \right\}^{1/2}$$

$$\leq \left\{ \int_0^1 \int_0^s (H(t) - [H]_{n,n})^2 ds\, dt \right\}^{1/2} \|x\| \quad . \tag{D.37}$$

Put

$$\epsilon_1 = \left\{ \int_0^1 \int_0^s (H(t) - [H]_{n,n})^2\, ds\, dt \right\}^{1/2} \quad .$$

Define $[y]_n$ to be the n^{th} partial sum of the Fourier series of y. Then for condition (C.5) put

$$\epsilon_2 = \|y - [y]_n\|_2 \quad .$$

From these definitions, then it follows that

$$\lim_{n\to\infty} \|x^* - x_N\|_2 = 0$$

or that the approximation converges in a root mean square sense.

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